Cálculo Numérico / Métodos Numéricos

Obtenção de raízes complexas
Método de Newton-Bairstow

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Obtenção de raízes complexas

- O método de Newton também pode ser usado para obter raízes complexas, utilizando aritmética complexa.
- Neste caso, veremos um método que obtém raízes complexas usando aritmética real.
- Se P(x) é um polinômio da forma:

$$P(x) = a^n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

e os coeficientes são reais, então as raízes complexas aparecem em pares conjugados, como solução de uma equação:

$$x^2 - \alpha x - \beta$$

Quociente e resto:

- Podemos expressar P(x) como:

$$P(x) = \left(x^2 - \alpha\, x - \beta\right) Q(x) + b_1(x - \alpha) + b_0$$

$$Q(x) = b_n\, x^{n-2} + b_{n-1}\, x^{n-3} + \ldots + b_2$$

- Obviamente, se α e β são raízes, b_0 e b_1 são iguais a zero.

- Vamos determinar quem são os coeficientes de Q(x). Multiplicamos Q(x) pelo termo quadrático e igualamos os coeficientes:

Igualando termos

$$P(x) = (x^2 - \alpha\, x - \beta)\, Q(x) + b_1(x - \alpha) + b_0$$

$$\begin{aligned} P(x) \;=\;& x^2\,(b_n\, x^{n-2} + b_{n-1}\, x^{n-3} + \ldots + b_2) \\ -\;& \alpha x\,(b_n\, x^{n-2} + b_{n-1}\, x^{n-3} + \ldots + b_2) \\ -\;& \beta\,(b_n\, x^{n-2} + b_{n-1}\, x^{n-3} + \ldots + b_2) \\ +\;& b_1(x - \alpha) + b_0 \end{aligned}$$

Rearrumando:

$$\begin{aligned} =\;& b_n\, x^n + (b_{n-1} - \alpha\, b_n)\, x^{n-1} + (b_{n-2} - \alpha\, b_{n-1} - \beta\, b_n)\, x^{n-2} \\ +\;& \ldots + (b_1 - \alpha\, b_2 - \beta\, b_3)\, x + (b_0 - \alpha\, b_1 - \beta\, b_2)\,. \end{aligned}$$

=

$$P(x) = a^n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

Termo a termo:

$$
\begin{aligned}
b_n &= a_n \,, \\
b_{n-1} &= a_{n-1} + \alpha\, b_n \,, \\
b_{n-2} &= a_{n-2} + \alpha\, b_{n-1} + \beta\, b_n \,, \\
&\vdots \\
b_1 &= a_1 + \alpha\, b_2 + \beta\, b_3 \,, \\
b_0 &= a_0 + \alpha\, b_1 + \beta\, b_2 \,.
\end{aligned}
$$

Como anteriormente, fazemos um "esquema prático" para cálculo:

	a_n	a_{n-1}	a_{n-2}	...	a_2	a_1	a_0
α	↓	+ αb_n	+ αb_{n-1}	...	+ αb_3	+ αb_2	+ αb_1
β	↓		+ βb_n	...	+ βb_4	+ βb_3	+ βb_2
	b_n	b_{n-1}	b_{n-2}	...	b_2	b_1	b_0

Sistema não linear

- O que queremos são valores de α e β que façam com que b_0 e b_1 se anulem.

$$\begin{cases} b_1\ (\alpha, \beta) & = & 0 \\ b_0\ (\alpha, \beta) & = & 0 \end{cases}$$

Note que b_0 e b_1 são funções de α e β.

Podemos resolver este sistema através do método de Newton para sistemas não lineares.

Lembrete: método de Newton para sistemas ñ-lineares

$$J(x_k, y_k)\begin{pmatrix} x_{k+1} - x_k \\ y_{k+1} - y_k \end{pmatrix} = \begin{pmatrix} -f(x_k, y_k) \\ -g(x_k, y_k) \end{pmatrix}$$

No nosso caso:

$$\delta\alpha_0 = \alpha_1 - \alpha_0 \quad \text{e} \quad \delta\beta_0 = \beta_1 - \beta_0$$

$$\begin{cases} \dfrac{\partial b_1}{\partial \alpha}\,\delta\alpha_0 + \dfrac{\partial b_1}{\partial \beta}\,\delta\beta_0 = -b_1(\alpha_0, \beta_0) \\[2ex] \dfrac{\partial b_0}{\partial \alpha}\,\delta\alpha_0 + \dfrac{\partial b_0}{\partial \beta}\,\delta\beta_0 = -b_0(\alpha_0, \beta_0) \end{cases}$$

Calculando as derivadas parciais (α)

$$
\begin{aligned}
b_n &= a_n \ , \\
b_{n-1} &= a_{n-1} + \alpha\, b_n \ , \\
b_{n-2} &= a_{n-2} + \alpha\, b_{n-1} + \beta\, b_n \ , \\
&\vdots \\
b_1 &= a_1 + \alpha\, b_2 + \beta\, b_3 \ , \\
b_0 &= a_0 + \alpha\, b_1 + \beta\, b_2 \ .
\end{aligned}
$$

$$\frac{\partial b_n}{\partial \alpha} = 0 \ ,$$

$$\frac{\partial b_{n-1}}{\partial \alpha} = b_n \ ,$$

$$\frac{\partial b_{n-2}}{\partial \alpha} = b_{n-1} + \alpha \frac{\partial b_{n-1}}{\partial \alpha} \ ,$$

$$\frac{\partial b_{n-3}}{\partial \alpha} = b_{n-2} + \alpha \frac{\partial b_{n-2}}{\partial \alpha} + \beta \frac{\partial b_{n-1}}{\partial \alpha} \ ,$$

$$\cdots\cdots$$

$$\frac{\partial b_1}{\partial \alpha} = b_2 + \alpha \frac{\partial b_2}{\partial \alpha} + \beta \frac{\partial b_3}{\partial \alpha} \ ,$$

$$\frac{\partial b_0}{\partial \alpha} = b_1 + \alpha \frac{\partial b_1}{\partial \alpha} + \beta \frac{\partial b_2}{\partial \alpha} \ .$$

Calculando as derivadas parciais (α)

$$
\begin{aligned}
b_n &= a_n \,, \\
b_{n-1} &= a_{n-1} + \alpha\, b_n \,, \\
b_{n-2} &= a_{n-2} + \alpha\, b_{n-1} + \beta\, b_n \,, \\
&\vdots \\
b_1 &= a_1 + \alpha\, b_2 + \beta\, b_3 \,, \\
b_0 &= a_0 + \alpha\, b_1 + \beta\, b_2 \,.
\end{aligned}
$$

$$\frac{\partial b_n}{\partial \alpha} = 0 \,,$$

$$\frac{\partial b_{n-1}}{\partial \alpha} = b_n \,,$$

c_n

$$\frac{\partial b_{n-2}}{\partial \alpha} = b_{n-1} + \alpha \frac{\partial b_{n-1}}{\partial \alpha} \,,$$

c_{n-1} c_n

$$\frac{\partial b_{n-3}}{\partial \alpha} = b_{n-2} + \alpha \frac{\partial b_{n-2}}{\partial \alpha} + \beta \frac{\partial b_{n-1}}{\partial \alpha} \,,$$

c_{n-2} c_{n-1} c_n

. . . . . .

$$\frac{\partial b_1}{\partial \alpha} = b_2 + \alpha \frac{\partial b_2}{\partial \alpha} + \beta \frac{\partial b_3}{\partial \alpha} \,,$$

c_2 c_3 c_4

$$\frac{\partial b_0}{\partial \alpha} = b_1 + \alpha \frac{\partial b_1}{\partial \alpha} + \beta \frac{\partial b_2}{\partial \alpha} \,.$$

c_1 c_2 c_3

Calculando os c_i's

$$
\begin{aligned}
c_n &= b_n \,, \\
c_{n-1} &= b_{n-1} + \alpha c_n \,, \\
c_{n-2} &= b_{n-2} + \alpha c_{n-1} + \beta c_n \,, \\
c_{n-3} &= b_{n-3} + \alpha c_{n-2} + \beta c_{n-1} \,, \\
&\vdots \\
c_2 &= b_2 + \alpha c_3 + \beta c_4 \,, \\
c_1 &= b_1 + \alpha c_2 + \beta c_3 \,.
\end{aligned}
$$

Procedimento prático aplicável

$$\frac{\partial b_n}{\partial \alpha} = 0 \,,$$

c_n

$$\frac{\partial b_{n-1}}{\partial \alpha} = b_n \,,$$

c_{n-1} , c_n

$$\frac{\partial b_{n-2}}{\partial \alpha} = b_{n-1} + \alpha \frac{\partial b_{n-1}}{\partial \alpha} \,,$$

c_{n-2} , c_{n-1} , c_n

$$\frac{\partial b_{n-3}}{\partial \alpha} = b_{n-2} + \alpha \frac{\partial b_{n-2}}{\partial \alpha} + \beta \frac{\partial b_{n-1}}{\partial \alpha} \,,$$

$\ldots\ldots$

c_2 , c_3 , c_4

$$\frac{\partial b_1}{\partial \alpha} = b_2 + \alpha \frac{\partial b_2}{\partial \alpha} + \beta \frac{\partial b_3}{\partial \alpha} \,,$$

c_1 , c_2 , c_3

$$\frac{\partial b_0}{\partial \alpha} = b_1 + \alpha \frac{\partial b_1}{\partial \alpha} + \beta \frac{\partial b_2}{\partial \alpha} \,.$$

Procedimento prático:

	a_n	a_{n-1}	a_{n-2}	...	a_2	a_1	a_0
α		$+\ \alpha b_n$	$+\ \alpha b_{n-1}$	...	$+\ \alpha b_3$	$+\ \alpha b_2$	$+\ \alpha b_1$
β			$+\ \beta b_n$	...	$+\ \beta b_4$	$+\ \beta b_3$	$+\ \beta b_2$
	b_n	b_{n-1}	b_{n-2}	...	b_2	b_1	b_0
α		αc_n	αc_{n-1}	...	αc_3	αc_2	
β			$+\ \beta c_n$	...	$+\ \beta c_4$	$+\ \beta c_3$	
	c_n	c_{n-1}	c_{n-2}	...	c_2	c_1	

$$c_2 \rightarrow \frac{\partial b_1}{\partial \alpha} \qquad c_1 \rightarrow \frac{\partial b_0}{\partial \alpha}$$

Por que estamos fazendo isso mesmo ?

$$\delta\alpha_0 = \alpha_1 - \alpha_0 \quad \text{e} \quad \delta\beta_0 = \beta_1 - \beta_0$$

c_2

$$\begin{cases} \dfrac{\partial b_1}{\partial \alpha}\,\delta\alpha_0 + \dfrac{\partial b_1}{\partial \beta}\,\delta\beta_0 = -b_1(\alpha_0, \beta_0) \\[2ex] \dfrac{\partial b_0}{\partial \alpha}\,\delta\alpha_0 + \dfrac{\partial b_0}{\partial \beta}\,\delta\beta_0 = -b_0(\alpha_0, \beta_0) \end{cases}$$

c_1

Ainda precisamos calcular as derivadas parciais em relação ao β

Calculando as derivadas parciais (β)

$$
\begin{aligned}
b_n &= a_n \,, \\
b_{n-1} &= a_{n-1} + \alpha\, b_n \,, \\
b_{n-2} &= a_{n-2} + \alpha\, b_{n-1} + \beta\, b_n \,, \\
&\vdots \\
b_1 &= a_1 + \alpha\, b_2 + \beta\, b_3 \,, \\
b_0 &= a_0 + \alpha\, b_1 + \beta\, b_2 \,.
\end{aligned}
$$

$$\frac{\partial b_n}{\partial \beta} = \frac{\partial b_{n-1}}{\partial \beta} = 0$$

$$\overset{c_n}{\frac{\partial b_{n-2}}{\partial \beta}} = bn \,,$$

$$\overset{c_{n-1}}{\frac{\partial b_{n-3}}{\partial \beta}} = b_{n-1} + \alpha \overset{c_n}{\frac{\partial b_{n-2}}{\partial \beta}} \,,$$

$$\overset{c_{n-2}}{\frac{\partial b_{n-4}}{\partial \beta}} = b_{n-2} + \alpha \overset{c_{n-1}}{\frac{\partial b_{n-3}}{\partial \beta}} + \overset{c_n}{\frac{\partial b_{n-2}}{\partial \beta}} \,,$$

$$\overset{c_3}{\frac{\partial b_1}{\partial \beta}} = b_3 + \alpha \overset{c_4}{\frac{\partial b_2}{\partial \beta}} + \beta \overset{c_5}{\frac{\partial b_3}{\partial \beta}} \,,$$

$$\overset{c_2}{\frac{\partial b_0}{\partial \beta}} = b_2 + \alpha \overset{c_3}{\frac{\partial b_1}{\partial \beta}} + \beta \overset{c_4}{\frac{\partial b_2}{\partial \beta}} \,.$$

Por que estamos fazendo isso mesmo ?

$$\delta\alpha_0 = \alpha_1 - \alpha_0 \quad \text{e} \quad \delta\beta_0 = \beta_1 - \beta_0$$

c_2 c_3

$$\begin{cases} \dfrac{\partial b_1}{\partial \alpha}\,\delta\alpha_0 + \dfrac{\partial b_1}{\partial \beta}\,\delta\beta_0 = -b_1(\alpha_0, \beta_0) \\[2ex] \dfrac{\partial b_0}{\partial \alpha}\,\delta\alpha_0 + \dfrac{\partial b_0}{\partial \beta}\,\delta\beta_0 = -b_0(\alpha_0, \beta_0) \end{cases}$$

c_1 c_2

$$\begin{cases} c_2\,\delta\alpha_0 + c_3\,\delta\beta_0 = -\,b_1(\alpha_0, \beta_0) \\ c_1\,\delta\alpha_0 + c_2\,\delta\beta_0 = -\,b_0(\alpha_0, \beta_0) \end{cases}$$

Exemplo

- Calcular duas raízes conjugadas da equação polinomial

$P(x) = x^4 - 2x^3 + 4x^2 - 4x + 4$

pelo método de Newton-Bairstow, iniciado em
$(\alpha_0, \beta_0) = (1,-1)$

Exemplo (solução)

Primeiramente calculamos os b_k e os c_k. Assim:

		1	-2	4	-4	4
	1		1	-1	2	-1
	-1			-1	1	-2
b_k's →		1	-1	2	-1	1
	1		1	0	1	
	-1			-1	0	
c_k's →		1	0	1	0	
		c_4	c_3	c_2	c_1	

$$\begin{cases} c_2\, \delta\alpha_0 + c_3\, \delta\beta_0 = -\, b_1(\alpha_0, \beta_0) \\ c_1\, \delta\alpha_0 + c_2\, \delta\beta_0 = -\, b_0(\alpha_0, \beta_0) \end{cases} \longrightarrow \begin{cases} 1.\ \delta\alpha_0 + 0.\ \delta\beta_0 = 1 \\ 0.\ \delta\alpha_0 + 1.\delta\beta_0 = -1 \end{cases}$$

Exemplo (solução)

$$\begin{cases} 1.\ \delta\alpha_0 + 0.\ \delta\beta_0 = 1 \\ 0.\ \delta\alpha_0 + 1.\delta\beta_0 = -1 \end{cases}$$

$\delta\alpha_0 = 1$ e $\delta\beta_0 = -1$.

$$\begin{aligned} \alpha_1 &= \alpha_0 + \delta\alpha_0 \Rightarrow \alpha_1 = 2\ , \\ \beta_1 &= \beta_0 + \delta\beta_0 \Rightarrow \beta_1 = -2\ . \end{aligned}$$

Repetindo o processo com os novos α e β:

	1	-2	4	-4	4
2		2	0	4	0
-2			-2	0	-4
	1	0	2	0	0

α e β acarretam raiz

Exemplo (solução)

Logo, $x^2 - \alpha\ x - \beta\ =\ x^2 - 2\ x + 2$ é um divisor exato de $P(x)$

$x = 1 \pm i$

$x = \pm \sqrt{2}\, i$

	1	-2	4	-4	4
2		2	0	4	0
-2			-2	0	-4
	1	0	2	0	0

$Q(x) = x^2+2$

α_1 e β_1 acarretam raiz