Eightfold Way From Dynamical First Principles in Strongly Coupled Lattice QCD

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We obtain from first principles, i.e. from the quark-gluon dynamics, the Gell'Mann-Ne'eman baryonic eightfold way in an imaginary-time functional integral formulation of strongly coupled lattice QCD in $3+1$ dimensions, with local $\mathrm{SU}(3)_c$ gauge and global $\mathrm{SU}(3)_f$ flavor symmetries. The form of the 56 baryon fields emerges naturally from the dynamics and is unveiled using the hyperplane decoupling method. There is *no* a priori guesswork. In the associated physical quantum-mechanical Hilbert space $\mathcal{H}$, spectral representations are derived for the two-baryon functions, which are used to rigorously detect the particles in the energy-momentum spectrum. Using the $\mathrm{SU}(3)_f$ symmetry, the 56 baryon states admit a decomposition into 8×2 states associated with a spin $1/2$ octet, and 10×4 states associated with a spin $3/2$ decuplet. The states are labelled by the quantum numbers of total hypercharge Y, total isospin I, its third component I_3 and the value of the quadratic Casimir of $\mathrm{SU}(3)_f$; they also carry a label of total spin J, and its z-component J_z. The total spin operators are defined using $\pi/2$ rotations about the spatial coordinate axes and for improper zero momentum baryon states agree with the infinitesimal generators of the continuum. The masses of the 56 baryon states have the form $M = -3\ln\kappa - 3\kappa^3/4 + \kappa^6 r(\kappa)$, with $r(\kappa)$ analytic. $r(\kappa) \equiv r_o(\kappa)$ is the same within the octet. For the decuplet, $r(\kappa) \equiv r_d(\kappa)$ and $r_d(0)$ is the same for each member; so that no mass splitting appears within the decuplet up to and including $\mathcal{O}(\kappa^6)$. However, we find an octet-decuplet mass splitting given by $[r_d(\kappa) - r_o(\kappa)]\kappa^6 = 3\kappa^6/4 + \mathcal{O}(\kappa^7)$. For spatial momentum $\vec{p} \neq 0$, $\vec{p} = (p^1, p^2, p^3) \in (-\pi, \pi]^3$, the 56 baryon dispersion curves have the form $w(\kappa, \vec{p}) = -3\ln\kappa - 3\kappa^3/4 + \kappa^3 \sum_{j=1,2,3} (1 - \cos p^j)/4 + r(\kappa, \vec{p})$, where $r(\kappa, \vec{p})$ is of $\mathcal{O}(\kappa^6)$. For the octet, $r(\kappa, \vec{p})$ is jointly analytic in κ and in each p^j, for small $|\mathcal{I}\mathrm{m}\, p^j|$. For each baryon, there is an anti-baryon related to it by charge conjugation and with identical spectral properties. It is shown that the spectrum associated with baryons and antibaryons is the only spectrum in the subspace of $\mathcal{H}$ with an odd number of quarks, up to near the meson-baryon energy threshold of $\approx -5\ln\kappa$. A new *time-reflection* is found which is used to define a local spin flip symmetry. The spin flip symmetry, together with the usual parity, time-reversal and spatial $\pi/2$ rotation symmetries and analytic implicit function arguments, are used to obtain these results. Our method extends to the $\mathrm{SU}(N)_f$ case to uncover $(2N+2)!/[3!(2N-1)!]$ baryon states, and also to treat mesons.

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I. INTRODUCTION

In 1963, M. Gell'Mann and Y. Ne'eman (see Ref. [1]) independently introduced the quark model based on three quark flavors (*up* u, *down* d and *strange* s) and the isospin or flavor symmetry $\mathrm{SU}(3)_f$ for the hadrons, that led to the eightfold way classification and which successfully predicted the existence of the Ω^- particle (see Refs. [2–5]). Later, O.W. Greenberg introduced in Ref. [6] the color quantum number in an ad hoc manner to give the right statistics properties for the Δ^{++} particle wave function in terms of the wave functions of its three constituent quarks, which are fermions. H. Fritsch, M. Gell'Mann and H. Leutwyler in Refs. [7, 9], and also Weinberg in Ref. [53], gave a more realistic and physical meaning for the quarks by introducing a color dynamics for them and providing a quantum field theoretical model for the strong interactions. This is the well known Quantum Chromodynamics model (QCD). It is a gauge model with a color local $\mathrm{SU}(3)_c$ symmetry and with eight gauge bosons (gluons) interacting with the quarks and among themselves. In Refs. [10, 11], QCD was soon seen to be asymptotically free for short distances (ultraviolet) if the number N_f of flavors is less than 17. Later, for both theoretical and experimental reasons, three other new quark flavors (*charm*, *bottom* and *top*) were incorporated to it.

In parallel, K. Wilson (see Ref. [12, 13] and references therein) introduced the lattice approximation which provides an ultraviolet cutoff, and developed a functional integral formalism for QCD on the lattice which was powerful enough to e.g. obtain the first results on the QCD particle spectrum and to exhibit confinement. Many other applications

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came soon after either within the strong coupling expansion or in other contexts (see e.g. Refs. [13–21] and the books given in Refs. [22, 23]). Numerical simulations on the lattice acquired an important status on the particle description and other questions which were not attainable using the traditional perturbation theory (see e.g Refs. [24–27]).

In the seventies, prospecting in a complementary direction, the works of E. Nelson, K. Symanzik, and the joint work of K. Osterwalder and R. Schrader set on rigorous basis the imaginary-time functional integral formalism for models with or without fermions. J. Glimm, A. Jaffe and, especially, T. Spencer [28] with his hyperplane decoupling method made it possible to determine the particle structure (spectrum) of quantum field models. (For more details about all this mathematical work, see the books in [29, 30] and references therein.) In this way, one was able to control the infinite-volume limit and to establish the gap and the upper gap properties for simple scalar models. K. Osterwalder and E. Seiler in Refs. [31, 32] provided the formalism to extend this analysis for models with fermions. More specifically, physical positivity was established, the physical Hilbert space and the energy-momentum operators were constructed and Feynman-Kac (F-K) formulae were obtained for correlations in lattice QCD models.

In spite of these important accomplishments, there is little known at the theoretical or mathematical level about the low-lying energy-momentum spectrum, especially on the existence of particles. Also, the connection between low-energy QCD and Nuclear Physics, which is ordinarily described in terms of effective baryon and meson fields, has not yet reached a satisfactory level (see e.g. Ref. [33]). For instance, the existence of nucleons and how they interact e.g. to form a stable state as the deuteron was not understood, from dynamical first principles (e.g. starting with a QCD model action), until recently.

In a series of papers of Refs. [34–42], adapting the hyperplane decoupling method to the lattice, we obtained the low-lying energy-momentum (E-M) spectrum of increasingly complex lattice QCD models, using an imaginary-time functional integral formulation, within the strong coupling regime (small hopping term). The one-hadron spectrum was rigorously obtained and the baryon and meson mass splitting was analyzed. The two-hadron bound-state spectrum was also obtained by considering a ladder approximation to a Bethe-Salpeter equation for the four-hadron correlation function on the lattice. Hadron-hadron bound states do not show up in simple one-flavor models, because of Pauli repulsion. Baryon-baryon and meson-meson bound states did appear in models with two-quark flavors for the first time. As it is done in Refs. [43, 44], it is worth mentioning that our method to determine bound states in strongly coupled lattice QCD is tuned up and includes the main ingredients to validate the ladder approximation results at a mathematically rigorous level.

It is a fundamental problem in particle physics to determine the low-lying E-M spectrum of QCD. Since there is no rigorous derivations of the particle spectrum from dynamics in a *more realistic* QCD model, we think it would be nice to provide one. Here, we do just that. We use our strategy and method to obtain the *baryon particle spectrum* for the three-flavor $\mathrm{SU}(3)_c$ lattice QCD model in $3+1$-dimensions, in the strong coupling regime (with hopping parameter $\kappa > 0$ and pure gauge coupling $\beta = 1/(2g_0^2)$ satisfying the condition $0 < \beta \ll \kappa \ll 1$).

As a main point in our work, new spectral representations are derived for two-baryon correlations, via Feyman-Kac formulas. These representations allow us to relate the complex-momentum singularities of the Fourier transforms of baryon correlations to the E-M spectrum, leading to particle dispersion curves and masses. We point out that associating decay rates of correlations with masses *without* spectral representations is meaningless, especially when spectral degeneracies are broken with small splittings.

It has become clear, from our method of obtaining the one-particle and the two-particle bound state spectra from the dynamics, that an important mathematical structure occurs. Many important and nice features are consequences of this structure. For instance, we emphasize that:

1. The particles (and their multiplicities), and the form of the corresponding fields that create them, are revealed in a simple direct way. There is *no* guesswork involved as to the form of the fields. For example, in the case of non-gauge invariant bosonic or fermionic models, our intuition directly guides us to the correct excitation creating fields. However, in the case of a system with a large number of internal degrees of freedom, and where a special role is played by composite fields, the form of the fields is not so obvious;

2. The excitations are associated with isolated dispersion curves (particles), i.e. the upper gap property. Furthermore, convergent expansions are obtained for the masses;

3. The particles that are revealed in the first item, and which are created by the one-particle fields, are the *only* particles of the model in the E-M spectrum, up to near the two-particle thresholds. Since these particles are gauge-invariant, confinement is confirmed up to near the two-particle threshold, meaning that only colorless states generate spectrum.

This mathematical structure is a "product" appearing in the calculation of certain derivatives of hyperplane decoupling parameters which are used to obtain many of the consequences mentioned above, as well as others.

In this work, we restrict our attention to the subspace $\mathcal{H}_o$ of the physical Hilbert space $\mathcal{H}$, generated by vectors with an *odd* number of quarks, which includes the baryons. In our method, the one-baryon fields that arise naturally from the quark-gluon dynamics, are local gauge-invariant composites of three tightly-bound quarks. They are specified by the quantum numbers of isospin and the *spin* labels of *each* of the three quarks which we refer to as the *individual basis*, of dimension 56. (Below, we explain the meaning of the term *spin* that we adopt for the lattice.) At this stage, and *without* using the structure of the global flavor symmetry $\mathrm{SU}(3)_f$, we already obtain 56 particles with asymptotic mass $-3\ln\kappa$.

We point out that in this lattice QCD model there are also mesons. Mesons lie in the *even* subspace $\mathcal{H}_e$ generated by vectors with an even number of quarks. Mesons are tightly bound quark-antiquark bound states and have asymptotic mass $-2\ln\kappa$. The meson sector is presently being treated in Ref. [45].

We show that the baryon particles are isolated from the rest of the spectrum up to near the meson-baryon threshold $\approx -5\ln\kappa$. We note that, by a naive counting argument, there are $216 = (3\times 2)^3$ states as each quark has three flavors and only two lower spin states among the four possible spin states. (For us, as explained below, the two upper spin states are related to antiparticles.) It is the gauge invariance of the unveiled excitation fields that reduces to 56 the number of linearly independent fields.

To make contact with the usual baryonic $\mathrm{SU}(3)_f$ Eightfold Way particle description, we perform a real orthogonal transformation from the individual basis to another basis called the *particle basis*. It turns out that, in the particle basis, the two-baryon matrix correlation G is closest to a diagonal matrix, and its elements are identified with the total spin 1/2 *octet* (of dimension 16) and the total spin 3/2 *decuplet* (of dimension 40) of baryons from the $\mathrm{SU}(3)_c\times\mathrm{SU}(3)_f$ eightfold way, i.e. the Gell'Mann-Ne'eman flavor eightfold way including color. In the particle basis, the baryon states are labelled by (and distinguished by this labelling) the total spin I, the third component I_3 of total isospin, the hypercharge Y, the value of the quadratic Casimir operator C_2, the total spin J and, finally, the z-component J_z of the total spin. (The cubic Casimir operator is not needed!) For each fixed total spin and z-component of total spin, the octet (decuplet) is a basis for the eight- (ten-) dimensional representation of $\mathrm{SU}(3)_f$.

Now we show how our spectral results recover the usual baryon particles in the Gell'Mann-Ne'eman eightfold way for QCD in the continuum. By $\mathrm{SU}(3)_f$ symmetry, for each of the two possible values of J_z, the octet baryon masses are all equal. The same is true for each of the four decuplet masses with fixed J_z. However, by the $\mathrm{SU}(3)_f$ *alone*, due to the lack of spatial rotation invariance SO(3) on the lattice forcing us to adopt different meaning for *spin* on the lattice as compared to the one in the continuum, we cannot conclude that the octet masses for the two J_z values are equal; nor that the decuplet masses are equal for the four J_z values. Of course, there is as well no way to relate the octet and the decuplet masses based solely on the flavor symmetry.

We find a *new* local symmetry $\mathcal{F}_s$, which we call *spin flip*, acting separately in the lower and in the upper spin components (see below). It is this new symmetry which allows us to show that the octet masses are all equal for $J_z = \pm 1/2$ and that the four decuplet masses are two-by-two equal (i.e. mass equality for $J_z = \pm 1/2$ and for $J_z = \pm 3/2$).

We remark that, for a Poincaré invariant quantum field theory, *if* the octet (decuplet) fields are states transforming according to the total spin $J = 1/2$ ($J = 3/2$) representation of mass m_o (m_d), then the equality of these masses is of course automatic by passing to the rest frame with a Lorentz boost and using SU(2) rotational invariance.

Here, we show that all the baryon masses have the form $-3\ln\kappa - 3\kappa^3/4 + r(\kappa)$, where $r(\kappa)$ is analytic. Moreover, up to and including order κ^6, the two (with fixed $|J_z|$) decuplet masses are equal. However, there is an $\mathcal{O}(\kappa^6)$ mass splitting between the octet and the decuplet states.

Let us go back to the baryon fields. Precisely speaking, the local, gauge-invariant (colorless) fields that naturally occur in our method in the individual basis and linear combinations of which create all baryon states can be written as

$$\bar{b}_{\vec{\alpha}\vec{f}} \equiv \epsilon_{a_1a_2a_3}\bar{\psi}_{a_1\alpha_1f_1}\bar{\psi}_{a_2\alpha_2f_2}\bar{\psi}_{a_3\alpha_3f_3}\,. \tag{1}$$

In the above, $\bar{\psi}_{a\alpha f}$ is a Grassmann quark field with gauge (color) index $a = 1,2,3$, α is a *lower* component Dirac spin index taking the values $\alpha = 3,4$ ($\alpha = 1,2$ are *upper* spin indices) and $f = 1,2,3 \equiv u,d,s$ is an $\mathrm{SU}(3)_f$ flavor index. ϵ_{abc} is the completely antisymmetric Levi-Civita symbol and makes $\bar{b}_{\vec{\alpha}\vec{f}}$ gauge-invariant. There are also Grassmann fields $\psi_{a\alpha f}$ in our model and, similarly to the $\bar{\psi}$'s of Eq. (1), and considering only upper spin components, create gauge-invariant antibaryon fields $b_{\vec{\alpha}\vec{f}}$.

As a fundamental point to understand our results, we now explain our use of the term *spin*. In the case of a Poincaré invariant theory, the field as well as the states have a definite transformation law under Poincaré transformations. By a boost, we pass to improper states in the rest frame which transform according to a representation of SU(2), the spatial rotation subgroup. The generators are taken as the spin operators which obey the usual angular momentum relations. $\pi/2$ rotations about the spatial coordinate axes are symmetries which we use to define the components of our lattice total angular momentum. For improper zero momentum baryon states obtained with local composite fields, which are expected to have zero spatial angular momentum, we define rectangular components of total spin which agree with the infinitesimal generators of rotations of the continuum. We also note that for zero hopping parameter,

$\kappa = 0$, there is an SU(4) symmetry which includes the SU(2) $\oplus$ SU(2) symmetry in the spin space of the model; for $\kappa > 0$ of course only the discrete rotation subgroup survive.

An important property of the gauge-invariant fields of Eq. (1) is that they are symmetric under the interchange of any two pairs $\alpha_i f_i \leftrightarrow \alpha_j f_j$. We refer to this property as the *totally symmetric property* (*tsp*).

Without *tsp*, there are $(2 \times 3)^3 = 216$ baryon fields; but if we count the number of them after imposing *tsp*, there are only 56 linearly independent ones [46]. From a purely group theoretic argument, the above composite field is a rank three symmetric tensor where each of the three identical factor spaces has dimension $2N_f$. As derived in Ref. [46], it is of dimension $(2N_f + 2)!/[3!(2N_f - 1)!]$ for SU$(N_f)_f$, and then 56 for SU$(3)_f$. In the particle basis, the 56 SU$(3)_f$ baryon fields are linear combinations of the individual basis fields. For example, for the spin 1/2 octet, the plus spin proton field is given by

$$p_+ = \frac{1}{3\sqrt{2}}\,\epsilon_{abc}(\bar\psi_{a+u}\bar\psi_{b-d} - \bar\psi_{a+d}\bar\psi_{b-u})\bar\psi_{c+u}\,,$$

and for the spin 3/2 decuplet the electric charge two Δ^{++} is given by

$$\Delta^{++} = \frac{1}{6}\,\epsilon_{abc}\bar\psi_{a+u}\bar\psi_{b+u}\bar\psi_{c+u}\,.$$

The fields for the other particles are given below in Section II C.

We now explain in some detail the product structure that arises in the hyperplane decoupling method and its importance not only for determining decay of correlations but also how it allows us to show the existence of baryon particles. Precisely, the analytic results we need are the following:

1. The exponential falloff of the elements of an appropriately chosen two-baryon correlation matrix G.

2. The faster decay of elements of the convolution inverse Γ of G.

3. The precise short-range behavior of Γ to low orders in κ. This, in turn, requires the knowledge of the short-range behavior of G.

The product structure plays an instrumental role in obtaining these results. For our rigorous one-particle analysis, we start with a generic two-particle correlation function $G_{LM}(x,y)$, for two *arbitrary* functions L and M with an odd number of elementary quark fields, and localized in the separated temporal hyperplanes x^0 and y^0, respectively. Supposing $x^0 < y^0$, for each pair $(p, p+1)$ of adjacent temporal hyperplanes that separate the two points x^0 and y^0, $x^0 \le p < p+1 \le y^0$, the hopping parameter κ, in the corresponding hopping term of the action, is substituted by a complex parameter κ_p. Note that setting $\kappa_p = 0$ disconnects the x^0 and y^0 hyperplanes since any link crossing the p-hyperplane is forbidden. Also, taking derivatives with respect to κ_p, at $\kappa_p = 0$, forces these links to be present.

It is important to stress that the correlation function $G_{LM}(x,y)$ is jointly analytic in all the κ_p's. Also, we note that the same procedure can also be applied to any other hyperplane for the space directions as well, but we will here concentrate on the time direction, as this will lead to the existence of particles and isolated dispersion curves (which we call the upper mass gap property).

By direct calculation, either because of imbalance in the number of quark fields or by integration over the interhyperplane gauge fields, the zeroth, the first and the second order derivatives with respect to κ_p do vanish at $\kappa_p = 0$. (This leads to a $\kappa^{3|x^0-y^0|}$ temporal decay for $G_{LM}(x,y)$.) For the third derivative, after performing a gauge integration over the gauge fields associated with three gauge bonds connecting the two hyperplanes, up to numerical factors, we obtain [the superscript (ℓ) denotes the ℓth derivative at $\kappa_p = 0$]

$$G^{(3)}_{LM}(x,y) = -\sum_{\vec\gamma,\vec g}\sum_{\vec w} G^{(0)}_{L\bar b_{\vec\gamma\vec g}}(x,(p,\vec w))\, G^{(0)}_{b_{\vec\gamma\vec g}M}((p+1,\vec w),y) + \ldots\,,$$

where only lower components occur in $\vec\gamma$. There is an additional term (omitted here) that is seen to involve antiparticle fields and that will vanish for our choice of the external particle fields $L(x)$ and $M(y)$.

We write the above expression schematically as

$$G^{(3)}_{LM}(x,y) = -[G^{(0)}_{L\bar b_{\vec\gamma\vec g}} \,\circ\, G^{(0)}_{b_{\vec\gamma\vec g}M}](x,y)\,,$$

and a similar expression holds for $x^0 > y^0$. Thus, on the rhs, the local, gauge-invariant, composite fields $\hat b_{\vec\gamma\vec g}$ have made their appearance, as well as a "product" of G's. What we want is a correlation function so that this relation is closed in the sense that the external field correlation function on the left is the same as the correlation functions on the right. This can be achieved taking $L = b_{\vec{\alpha} f}$ and $M = \bar{b}_{\vec{\beta}\vec{h}}$ so that, upon letting G denote this correlation function, we have

$$G^{(3)}(x,y) = -[G^{(0)} \circ G^{(0)}](x,y)\,,$$

which we refer to as the *product structure* [see Eq. (14) below].

As described above, by eliminating linear dependencies, we reduce the dimension from 216 to 56, which enables us to define the convolution inverse Γ in the reduced space. We remark that, up to now, *no* structure of the global flavor symmetry group $SU(3)_f$ has been used.

As for G, by considering the κ_p derivatives of the relation $\Gamma\, G = 1 = G\,\Gamma$, it is can be seen that, for temporally separated points, $\Gamma^{(r)}(x,y) = 0$, $r = 0, 1, 2$ and,

$$\Gamma^{(3)}(x,y) = -[\Gamma^{(0)} G^{(3)} \Gamma^{(0)}](x,y)\,.$$

Here is where the product structure enters. Using the product formula for $G^{(3)}$, shows that $\Gamma^{(3)} = 0$, for temporal separation greater than one. This in turn leads to a faster temporal decay of Γ as compared to G. As a result, by the Payley-Wiener theorem (see Ref. [47]), the Fourier transform [$p = (p^0, \vec{p})$, with $p^0 \in (-\pi, \pi]$ and $\vec{p} = (p^1, p^2, p^3) \in (-\pi, \pi]^3$ are the conjugate variables] $\tilde{\Gamma}(p) \equiv \tilde{\Gamma}(p^0, \vec{p})$ is analytic in a larger strip $|\mathcal{I}\mathrm{m}\, p^0| < -4 \ln \kappa$, as compared to the analyticity domain $|\mathcal{I}\mathrm{m}\, p^0| < -3 \ln \kappa$, for $\tilde{G}(p^0, \vec{p})$. It turns out that $\Gamma^{(4)} = 0$ so that we have analyticity of $\tilde{\Gamma}(p)$ in an even larger strip $|\mathcal{I}\mathrm{m}\, p^0| < -5 \ln \kappa$.

Using the Feynman-Kac formula, $\tilde{G}(p)$ admits a spectral representation and, from this representation, it is seen that singularities on the imaginary p^0 axis are points in the E-M spectrum. A consequence of the above discussion, since $\tilde{\Gamma}(p)\tilde{G}(p) = 1$, is that

$$\tilde{\Gamma}^{-1}(p) = \frac{\{\mathrm{cof}\,[\tilde{\Gamma}(p)]\}^t}{\det \tilde{\Gamma}(p)}\,,$$

for fixed $\vec{p}$ and κ, gives a meromorphic extension of $\tilde{G}(p)$, in p^0, and the dispersion curves $w(\vec{p})$ satisfy the equation

$$\det \tilde{\Gamma}(p^0 = iw(\vec{p}), \vec{p}) = 0\,.$$

The computation of the above determinant and the determination of the solutions $w(\vec{p})$ is simplified by going to the most diagonal form of $\tilde{\Gamma}$ as possible, which occurs in the particle basis.

It is here that we use the isospin $SU(3)_f$ symmetry of the model. In the particle basis (recall it is related to the individual basis by a real orthogonal transformation) the matrix $\tilde{G}$ decomposes into eight identical 2×2 blocks and ten identical 4×4 blocks, and $\tilde{\Gamma}$ has the same block structure. Noting that the product structure is preserved under an orthogonal transformation shows that Γ also has the same faster temporal decay in the particle basis as in the individual basis. Consequently, $\tilde{\Gamma}(p^0, \vec{p})$ has a larger strip analyticity.

We remark that, although the isospin symmetry leads to a simpler block structure for $\tilde{\Gamma}$, the use of only $SU(3)_f$ symmetry does not ensure a simplification within each block, and for this we appeal to other symmetries. By using spin flip symmetry $\mathcal{F}_s$, time reversal $\mathcal{T}$, spatial rotations of $\pi/2$ about the third axis and parity $\mathcal{P}$, the 2×2 blocks are diagonal and multiples of the identity. However, we still do not know if the 4×4 blocks are diagonal for $\vec{p} \neq \vec{0}$. This is the best we can do and our analysis of dispersion curves by the methods we employ here is more limited for the decuplet sector as compared with the octet. The 4×4 blocks turn out to be diagonal at $\vec{p} = \vec{0}$ as explained next.

For the masses ($\vec{p} = \vec{0}$), the use of the global flavor symmetry $SU(3)_f$, $\mathcal{T}\mathcal{P}$ (composition of parity $\mathcal{P}$ and time reversal $\mathcal{T}$), and $\pi/2$ rotation about the third spatial axis is enough to show that $\tilde{\Gamma}(p^0 = iM, \vec{p} = \vec{0})$ is diagonal. The determinant factorizes and the 56 mass determining equations are

$$\tilde{\Gamma}_{rr}(p^0 = iM, \vec{p} = \vec{0}) = 0\,,$$

where r are collective baryon labelling indices.

Using the additional spin flip $\mathcal{F}_s$ symmetry shows that the equations are identical for the octet baryon and there are two distinct equations for the decuplets: one for absolute value of total spin third component 3/2 and one for 1/2. The equations are solved by using an auxiliary function method and, after making a nonlinear transformation in the p^0 variable, the problem can be cast into the application of the analytic implicit function theorem.

Coming back to dispersion curves, we find that the determinant of $\tilde{\Gamma}(p)$ factorizes in the product of the determinants of the multiple of identity 2×2 octet blocks (and then the product of its diagonal elements) and the determinants of the 4×4 spin 3/2 decuplet blocks. Again, as was done for the masses, the single dispersion curve for the octet is solved by the auxiliary field method and the analytic implicit function theorem. For the 4×4 block, the determinant factorizes. Using the spin flip symmetry, only two possibly distinct equations occur for the dispersion curves. We show that up to and including order κ^6 it does not occur, but maybe it occurs at a higher order in κ. By using a Rouché theorem argument (see Refs. [30, 56]), we show that, for fixed κ and $\vec{p}$, there are precisely four solutions $w_i(\vec{p})$, pairwise identical and may differ at order κ^6. Considering all $\vec{p} \in [-\pi, \pi)^3$, Due to analytical difficulties, we have not been able to show these solutions, for each $\vec{p}$, give rise to curves, but we keep using this terminology for simplicity.

With all this, concerning the octet and the decuplet, we show that all 56 dispersion *curves* have the form

$$w(\vec{p}) = -3 \ln \kappa - 3\kappa^3/4 + p_\ell^2 \kappa^3/8 + r(\kappa, \vec{p}) \,,$$

where $p_\ell^2 = 2\sum_{i=1,2,3}(1 - \cos p^i)$. For the octet, $r(\kappa, \vec{p})$ is jointly analytic in κ and in each spatial momentum component p^i, $i = 1, 2, 3$, for small $|\vec{p}|$. For both the octet and the decuplet, $r(\kappa, \vec{p})$ is of order κ^6.

With the methods described above, we have shown the existence of one-baryon states in the one-baryon subspace $\mathcal{H}_b \subset \mathcal{H}_o$, generated by vectors of the form $\bar{b}_{\vec{\alpha}\vec{f}}(x)$. Another choice of L and M in the product structure formula leads to antibaryons, generated by $b_{\vec{\alpha}\vec{f}}(x)$, $\vec{\alpha}$ with all upper indices, and related to baryons by charge conjugation $\mathcal{C}$. Consequently, baryons and antibaryons have identical spectral properties.

Our method extends directly to treat the even subspace $\mathcal{H}_e \subset \mathcal{H}$ (generated by vectors with an even number of Fermi fields) where the mesons lie. The analysis of the mesonic sector showing the existence of the eightfold way particles will appear soon in Ref. [45]. The general form of the local, gauge-invariant, composite meson fields is a linear combination of the basic excitation creating fields const $\bar{\psi}_{a,\alpha,f_1}\psi_{a,\beta,f_2}$, where α (β) is a lower (upper) component. There are a total of $(2 \times 3)^2 = 36$ linearly independent fields. Linear combinations of these fields give rise to the flavor singlet and octet of pseudo-scalar and vector eightfold way mesons.

Even though we have shown the existence of baryons in $\mathcal{H}_b$ we need to do more to show that these baryon states generate all the spectrum in $\mathcal{H}_o$ up to near the meson-baryon threshold $\simeq -5 \ln \kappa$. This is what we call the upper gap property in $\mathcal{H}_o$. Our results do not exclude spectrum, in $(0, -5\ln \kappa)$, generated by other vectors in $\mathcal{H}_o$. To show that the baryon dispersion curves are isolated up to near the meson-baryon threshold, in all $\mathcal{H}_o$, and that the spectrum associated with the 56 baryon and 56 antibaryon states is the only spectrum up to the meson-baryon threshold, we adapt a subtraction method from Ref. [35].

The paper is organized as follows. In Section II A, we define the model, physical Hilbert space, energy-momentum operators, give the Feynman-Kac formula and obtain the approximate one-baryon spectrum. Our main results are stated in Theorem 1 . The rest of the paper deals with the proof of Theorem 1 , which follows from three additional theorems. Theorems 2 and 3 are stated in Section II B, using the individual basis, and give global upper bounds on the decay of the two-baryon function and its convolution inverse. In Section II C, we introduce the particle basis and state Theorem 4 on the short distance behavior of these quantities. In Section III, we use $\mathrm{SU}(3)_f$ symmetry and the Feynman-Kac formula to define the Hilbert space operators for hypercharge and isospin, which were used in labelling the baryon state vectors of Section II C. In addition, we treat the spin flip symmetry $\mathcal{F}_s$ and show it is implemented by an anti-unitary operator in the Hilbert space $\mathcal{H}$. Together with the material discussed in three Appendices, the analysis up to this point allows us to establish our results in the one-baryon sector of $\mathcal{H}$ and, in Section IV, by establishing a suitable subtraction method, we show that the baryon, antibaryon spectrum is the only spectrum up to near the meson-baryon threshold (upper gap property). In Section 4, we make some final remarks. Appendix A is devoted to the analysis of symmetries and its use to obtain important correlation function identities and relations which are used to derive the one-particle spectrum. In Appendix B, we prove the global decay bounds and the short distance behavior of the two-baryon function G and its convolution inverse Γ. To obtain the short-distance behavior, we derive a general formula for calculating $G(x, y)$ restricted only to non self-intersecting path contributions connecting x to y. Lastly, in Appendix C, we establish the relationship between the particles fields revealed in Section II C and the non-dynamical group theoretical construction based on the decomposition $3 \times 3 \times 3 = 10 \oplus 8 \oplus 8 \oplus 1$ of the $3 \times 3 \times 3$ representation of $\mathrm{SU}(3)_f$ into irreducible representations.

Throughout the paper the symbol $\mathcal{O}(1)$ denotes a positive constant the value of which is irrelevant for our purposes.

II. MODEL AND RESULTS

A. The Model, the Physical Hilbert Space, Energy-Momentum Operators and Feynman-Kac Formula

We now introduce our $\mathrm{SU}(3)_c$, $d+1$ dimensional, imaginary-time lattice QCD model, where $d = 3$ is the space dimension. The partition function is given formally by $Z = \int e^{-S(\psi,\bar{\psi},g)}\, d\psi\, d\bar{\psi}\, d\mu(g)$, and for a function $F(\bar{\psi}, \psi, g)$, the normalized correlations are denoted by

$$\langle F\rangle \;=\; \frac{1}{Z}\int\, F(\bar\psi,\psi,g)\, e^{-S(\psi,\bar\psi,g)}\; d\psi\, d\bar\psi\, d\mu(g)\,. \tag{2}$$

The model action $S\equiv S(\psi,\bar\psi,g)$ is Wilson's action (see Ref. [13])

$$\begin{aligned} S \;=\; & \frac{\kappa}{2}\sum \bar\psi_{a,\alpha,f}(u)\Gamma^{\sigma e^\mu}_{\alpha\beta}\,(g_{u,u+\sigma e^\mu})_{ab}\psi_{b,\beta,f}(u+\sigma e^\mu)\\ & + \sum_{u\in\mathbb{Z}^4_o} \bar\psi_{a,\alpha,f}(u)M_{\alpha\beta}\psi_{a,\beta,f}(u) - \frac{1}{g_0^2}\sum_p \chi(g_p)\,, \end{aligned} \tag{3}$$

where, besides the sum over repeated indices $\alpha,\beta=1,2,3,4$ (spin), $a=1,2,3$ (color) and $f=1,2,3\equiv u,d,s$ (isospin), the first sum runs over $u=(u^0,\vec u)=(u^0,u^1,u^2,u^3)\in\mathbb{Z}^4_o\equiv\{\pm1/2,\pm3/2,\pm5/2...\}\times\mathbb{Z}^3$, $\sigma=\pm1$ and $\mu=0,1,2,3$. Here, we are adopting the label 0 for the time direction and e^μ, $\mu=0,1,2,3$, denotes the unit lattice vector for the μ-direction, and the direction described by e^3 will also be called the z-direction. The choice of the shifted lattice for the time direction, avoiding the zero-time coordinate, is so that, in the continuum limit, two-sided equal time limits of quark Fermi fields correlations can be accommodated. At each site $u\in\mathbb{Z}^4_o$, there are fermionic Grassmann fields $\psi_{a\alpha f}(u)$, associated with a quark, and fields $\bar\psi_{a\alpha f}(u)$, associated with an antiquark, which carry a Dirac spin index α, an $\mathrm{SU}(3)_c$ color index a and isospin f. The meaning of the term *spin* we adopt for the lattice is given in Section I, and we refer to $\alpha=1,2$ as *upper* spin indices and $\alpha=3,4$ or $+$ or $-$ respectively, as *lower* ones. For each nearest neighbor oriented lattice bond $<u,u\pm e^\mu>$ there is an $\mathrm{SU}(3)_c$ matrix $U(g_{u,u\pm e^\mu})$ parametrized by the gauge group element $g_{u,u\pm e^\mu}$ and satisfying $U(g_{u,u+e^\mu})^{-1}=U(g_{u+e^\mu,u})$. Associated with each lattice oriented plaquette p there is a plaquette variable $\chi(U(g_p))$ where $U(g_p)$ is the orientation-ordered product of matrices of $\mathrm{SU}(3)_c$ of the plaquette oriented bonds, and χ is the real part of the trace. For notational simplicity, we sometimes drop U from $U(g)$. Concerning the parameters, we take the quark-gauge coupling or hopping parameter $\kappa>0$. Also, $g_0>0$ describes the pure gauge strength and $M\equiv M(m,\kappa)=(m+2\kappa)I_4$, I_n being the $n\times n$ identity matrix. Given κ, for simplicity and without loss of generality, $m>0$ is chosen such that $M_{\alpha\beta}=\delta_{\alpha\beta}$, meaning that $m+2\kappa=1$. Also, we take $\Gamma^{\pm e^\mu}=-I_4\pm\gamma^\mu$, where γ_μ, satisfying $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}I_4$, are the 4×4 hermitian traceless anti-commuting Dirac matrices

$$\gamma^0=\begin{pmatrix} I_2 & 0\\ 0 & -I_2\end{pmatrix} \quad ; \quad \gamma^j=\begin{pmatrix} 0 & i\sigma^j\\ -i\sigma^j & 0\end{pmatrix}\,,$$

where σ^j, $j=1,2,3$, denotes the hermitian traceless anti-commuting 2×2 Pauli spin matrices. The measure $d\mu(g)$ is the product measure over non-oriented bonds of normalized $\mathrm{SU}(3)_c$ Haar measures (see Ref. [48]). There is only one integration variable per bond, so that g_{uv} and g^{-1}_{vu} are not treated as distinct integration variables. The integrals over Grassmann fields are defined according to Ref. [49]. For a polynomial in the Grassmann variables with coefficients depending on the gauge variables, the fermionic integral is defined as the coefficient of the monomial of maximum degree, i.e. of $\prod_{u,\ell}\bar\psi_\ell(u)\psi_\ell(u)$, $\ell\equiv(\alpha,a,f)$. In Eq. (2), $d\psi\,d\bar\psi$ means $\prod_{u,\ell}d\psi_\ell(u)\,d\bar\psi_\ell(u)$ such that, with a normalization $\mathcal{N}_1=\langle1\rangle$, we have $\langle\psi_{\ell_1}(x)\,\bar\psi_{\ell_2}(y)\rangle \;=\; (1/\mathcal{N}_1)\;\int\,\psi_{\ell_1}(x)\,\bar\psi_{\ell_2}(y)\,e^{-\sum_{u,\ell_3,\ell_4}\bar\psi_{\ell_3}(u)O_{\ell_3\ell_4}\psi_{\ell_4}(u)}\,d\psi\,d\bar\psi$ $=O^{-1}_{\alpha_1,\alpha_2}\delta_{a_1a_2}\delta_{f_1f_2}\delta(x-y)$, with a Kronecker delta for space-time coordinates, and where O is diagonal in the color and isospin indices. In our model, for $\kappa=0$, we have the two-point functions $\langle\psi_{\ell_1}(x)\,\bar\psi_{\ell_2}(y)\rangle=\delta_{\alpha_1,\alpha_2}\delta_{a_1a_2}\delta_{f_1f_2}\delta(x-y)$, and the integral of monomials is given by Wicks theorem.

Throughout the paper, we work in the strong coupling regime i.e. we take $0<g_0^{-1}\ll\kappa\ll1$, $m=1-2\kappa\lesssim1$. With this restriction on the parameters, and since we have chosen $\Gamma^{\pm e^\mu}$ matrices within the two-parameter family described in Ref. [32], positivity is preserved and there is a quantum mechanical Hilbert space $\mathcal{H}$ of physical states (see below). Furthermore, the condition $m>0$ guarantees that the one-particle free Fermion dispersion curve increases in each positive momentum component and is convex for small momenta.

The action of Eq. (3) is invariant by the gauge transformations given by, for $x\in\mathbb{Z}^4_o$ and $h(x)\in\mathrm{SU}(3)_c$,

$$\begin{aligned} \psi(x) \;&\mapsto\; h(x)\,\psi(x)\,,\\ \bar\psi(x) \;&\mapsto\; \bar\psi(x)\,[h(x)]^{-1}\,,\\ U(g_{x+e^\mu,x}) \;&\mapsto\; h(x+e^\mu)\,U(g_{x+e^\mu,x})\,[h(x)]^{-1}\,. \end{aligned} \tag{4}$$

Concerning the global $\mathrm{SU}(3)_f$ isospin symmetry of the action, we follow the treatment given in Ref. [40]. We treat isospin symmetry at the level of correlation functions in Appendix A and implement them as physical Hilbert space operators in Section III. Other symmetries of the action of Eq. (3), such as time reversal, charge conjugation, parity, coordinate reflections and spatial rotations, which can be implemented by unitary (anti-unitary for time reversal) operators on the physical Hilbert space $\mathcal{H}$, were treated in Ref. [35] and do *not* affect the isospin indices. Furthermore, in Section III and in Appendix A, a new *time reflection* symmetry, to be distinguished from time reversal, of the correlation functions is used to define a new local spin flip symmetry which is shown to be useful for obtaining additional relations between correlations. We also define the rectangular components of a lattice total angular momentum and spin operators. In Section III, we also show how to implement these symmetry transformations as operators in the physical Hilbert space.

By polymer expansion methods (see Refs. [30, 32]), the thermodynamic limit of correlations exists and truncated correlations have exponential tree decay. The limiting correlation functions are lattice translational invariant. Furthermore, the correlation functions extend to analytic functions in the global coupling parameters κ and $\beta = 1/(2g_0^2)$ and also in any finite number of local coupling parameters. For the formal hopping parameter expansion, see Refs. [22, 23, 31, 32].

The underlying quantum mechanical physical Hilbert space $\mathcal{H}$ and the E-M operators H and P^j, $j = 1, 2, 3$ are defined as in Refs. [35, 36]. We start from gauge invariant correlations, with support restricted to $u^0 = 1/2$ and we let $T_0^{x^0}$, $T_i^{x^i}$, $i = 1, 2, 3$, denote translation of the functions of Grassmann and gauge variables by $x^0 \geq 0$, $\vec{x} = (x^1, x^2, x^3) \in \mathbb{Z}^3$. For F and G only depending on coordinates with $u^0 = 1/2$, we have the F-K formula

$$(G, \check{T}_0^{x^0} \check{T}_1^{x^1} \check{T}_2^{x^2} \check{T}_3^{x^3} F)_{\mathcal{H}} = \langle [T_0^{x^0} \vec{T}^{\vec{x}} F] \Theta G \rangle \ , \tag{5}$$

where $T^{\vec{x}} = T_1^{x^1} T_2^{x^2} T_3^{x^3}$ and Θ is an anti-linear operator which involves time reflection. Following Ref. [32], the action of Θ on single fields is given by

$$\begin{aligned} \Theta \bar{\psi}_{a\alpha}(u) &= (\gamma^0)_{\alpha\beta} \psi_{a\beta}(tu) \,, \\ \Theta \psi_{a\alpha}(u) &= \bar{\psi}_{a\beta}(tu) (\gamma_0)_{\beta\alpha} \,; \end{aligned}$$

where $t(u^0, \vec{u}) = (-u^0, \vec{u})$, for A and B monomials, $\Theta(AB) = \Theta(B)\Theta(A)$; and for a function of the gauge fields $\Theta f(\{g_{uv}\}) = f^*(\{g_{(tu)(tv)}\})$, $u, v \in \mathbb{Z}_o^{d+1}$, where $*$ means complex conjugate. Θ extends anti-linearly to the algebra. For simplicity, we do not distinguish between Grassmann, gauge variables and their associated Hilbert space vectors in our notation. As linear operators in $\mathcal{H}$, $\check{T}_\mu$, $\mu = 0, 1, 2, 3$, are mutually commuting; $\check{T}_0$ is self-adjoint, with $-1 \leq \check{T}_0 \leq 1$, and $\check{T}_{j=1,2,3}$ are unitary. So, we write $\check{T}_j = e^{iP^j}$ and $\vec{P} = (P^1, P^2, P^3)$ is the self-adjoint momentum operator. Its spectral points are $\vec{p} \in \mathbf{T}^3 \equiv (-\pi, \pi]^3$. Since $\check{T}_0^2 \geq 0$, the energy operator $H \geq 0$ can be defined by $\check{T}_0^2 = e^{-2H}$. We call a point in the E-M spectrum associated with spatial momentum $\vec{p} = \vec{0}$ a mass and, to be used below, we let $\mathcal{E}(\lambda^0, \vec{\lambda})$ be the product of the spectral families of $\check{T}_0$, P^1, P^2 and P^3. By the spectral theorem (see Ref. [50], we have

$$\check{T}_0 = \int_{-1}^{1} \lambda^0 dE_0(\lambda^0) \quad , \quad \check{T}_{j=1,2,3} = \int_{-\pi}^{\pi} e^{i\lambda^j} dF_j(\lambda^j) \,,$$

so that $\mathcal{E}(\lambda^0, \vec{\lambda}) = E_0(\lambda^0) \prod_1^3 F_j(\lambda^j)$. The positivity condition $\langle F \Theta F \rangle \geq 0$ is established in Ref. [32], but there may be nonzero F's such that $\langle F \Theta F \rangle = 0$. If the collection of such F's is denoted by $\mathcal{N}$, a pre-Hilbert space $\mathcal{H}'$ can be constructed from the inner product $\langle G \Theta F \rangle$ and the physical Hilbert space $\mathcal{H}$ is the completion of the quotient space $\mathcal{H}'/\mathcal{N}$, including also the cartesian product of the inner space sectors, the color space $\mathbb{C}^3$, the spin space $\mathbb{C}^4$ and the isospin space $\mathbb{C}^3$.

Here, we analyze the one-baryon sector of $\mathcal{H}_o \subset \mathcal{H}$, the subspace with an *odd* number of fermion fields (quarks). The mesons lie in the *even* subspace $\mathcal{H}_e$. Points in the E-M spectrum are detected as singularities in momentum space spectral representations of suitable two-baryon field correlations given below.

In order to classify and label the baryon states, using the flavor symmetry $\mathrm{SU}(3)_f$ (which holds for any κ!) we define, as usual, the total isospin I, third component of total isospin I_3 and total hypercharge Y operators associated with the SU(2) and, respectively, U(1) subgroups of $\mathrm{SU}(3)_f$. Together with the value of the quadratic Casimir C_2, the quantum numbers of isospin and hypercharge are used to label the baryon fields. Also, we use the labels of total spin and its z-component, J and J_z. This labelling distinguishes between all these baryon states and there is no need for the cubic Casimir C_3! The construction of all these operators is given in Section III, and we now come to the point where our results can be stated. The main results of this paper are given in the theorem below.

Theorem 1 *The energy-momentum spectrum of lattice QCD with three quark flavors u, d and s, and global flavor symmetry* $\mathrm{SU}(3)_f$*, as defined by the action given in Eq. (3), in the strong coupling regime* $0 < \beta \ll \kappa \ll 1$*, in the odd sector* $\mathcal{H}_o$ *of the physical Hilbert space* $\mathcal{H}$*, generated by vectors with an odd number of quark fields, and up to near the meson-baryon threshold of* $-(5-\epsilon)\ln\kappa$*,* $0 < \epsilon \ll 1$*, is generated by* 56 *baryon particles which are strongly bound bound states with three quarks. Also, for each baryon, there is a corresponding antibaryon, related by charge conjugation, and with identical spectral properties. These* 56 *baryon states are organized into two octets of total spin* $J = 1/2$ *and* *with* $C_2 = 3$, *and four decuplets of total spin* $J = 3/2$ *and with* $C_2 = 6$. *There is one octet associated with the third component of total spin* $J_z = +1/2$ *and one associated with* $J_z = -1/2$. *Also, there is one decuplet associated with each value* $J_z = \pm 3/2, \pm 1/2$. *All the masses of the baryons (i.e. the energy-momentum spectrum at* $\vec{p} = \vec{0}$*) in the octet and decuplet have the form*

$$M \equiv M(\kappa) = -3\ln\kappa - 3\kappa^3/4 + \kappa^6 r(\kappa)\,,$$

where $r(\kappa)$ *is analytic and* $r(0) \neq 0$. $r(\kappa)$ *is the same for all members of the octets, and we let* $r_o(0)$ *denote* $r(0)$ *for this case. For the decuplets,* $r(\kappa)$ *only depends on* $|J_z|$. *For all members of the decuplets,* $r(0) = r_d(0)$. *There is a mass splitting between the octets and the decuplets given by* $[r_d(0) - r_o(0)]\kappa^6 = 3\kappa^6/4$. *If there is mass splitting within the decuplets, it is of order* κ^7 *or higher.*

The energy-momentum spectrum for $\vec{p} \neq \vec{0}$ *is associated with baryons and is given by* 56 *dispersion relations, which all have the form*

$$w(\vec{p}) \equiv w(\kappa,\vec{p}) = \left[-3\ln\kappa - 3\kappa^3/4 + p_\ell^2\kappa^3/8\right] + r(\kappa,\vec{p})\,,$$

where $p_\ell^2 \equiv 2\sum_{i=1}^{3}(1-\cos p^i)$, *and* $p^{i=1,2,3} \in [-\pi,\pi)$ *are the spatial momentum components. For the octet,* $r(\kappa,\vec{p}) = \kappa^6 r_0(\kappa,\vec{p})$, *where* $r_0(\kappa,\vec{p})$ *is jointly analytic in* κ *and in each* p^i, $i = 1,2,3$, *for* $|\mathcal{I}\mathrm{m}\, p^j|$ *small. Also, for the octets, the curves* $w(\vec{p})$ *are convex for small* $|\vec{p}|$. *For the members of each decuplet,* $r(\kappa,\vec{p})$ *is of order* κ^6. □

Proof of Theorem 1 : Theorem 1 follows from Theorems 2 , 3 and 4 and the analysis given in the next two subsections and in three appendices. The upper gap property results from the subtraction method given in Section IV.

■

B. Individual Basis and Approximate One-Baryon Spectrum

To determine the form of the baryon fields and their correlations, following the strategy outlined in the introduction, using the hyperplane decoupling method, we start by considering the functions $L, M \in \mathcal{H}_o$. With $\check{T}^{\vec{x}} = \check{T}_1^{x^1}\check{T}_2^{x^2}\check{T}_3^{x^3}$, we have, for $|u^0 - v^0| \geq 1$,

$$\left(L, \check{T}_0^{|u^0-v^0|-1}\check{T}^{\vec{v}-\vec{u}}M\right)_{\mathcal{H}} = \left\langle [T_0^{|u^0-v^0|-1}T^{\vec{v}-\vec{u}}M]\,\Theta L\right\rangle\,. \tag{6}$$

For $v^0 > u^0$,

$$\langle M(v-u-e^0)\Theta L\rangle = -\langle \Theta L M(v-u-e^0)\rangle = -\langle F(u)M(v)\rangle\,,$$

where $F(u) = \Theta L(u+e^0)$. For $v^0 < u^0$, moving first the E-M operators to the lhs and then taking the complex conjugate, we get

$$\left(M, \check{T}_0^{|u^0-v^0|-1}\check{T}^{\vec{v}-\vec{u}}L\right)^*_{\mathcal{H}} = \left\langle [T_0^{|u^0-v^0|-1}T^{\vec{u}-\vec{v}}L]\Theta M\right\rangle^* = \langle L(u-v-e^0)\Theta M\rangle = \langle L(u)H(v)\rangle^*\,, \tag{7}$$

where $H(v) = \Theta M(v+e^0)$.

Next, we consider the hyperplane derivatives of the above correlation with respect to κ_p, at $\kappa_p = 0$. The r^{th} derivatives $r = 0,2,4$ are zero by imbalance of fermions. The first derivative also gives zero by interhyperplane gauge field integration. Thus, for $|u^0 - v^0| > 0$,

$$\langle F(u)M(v)\rangle^{(r=0,1,2,4)} = 0\,.$$

To analyze the third derivative, using the integral $\mathcal{I}_3$ of Eq. (B2) to perform the interhyperplane gauge field integration which gives rise to Levi-Civita symbols, with all fields at the same point, we let

$$\hat{b}_{\vec{\alpha}\vec{f}} = \epsilon_{abc}\,\hat{\psi}_{a\alpha_1 f_1}\hat{\psi}_{b\alpha_2 f_2}\hat{\psi}_{c\alpha_3 f_3}\,,$$

and use the superscript u (ℓ) to denote that only upper, $\alpha_i = 1,2$ (lower, $\alpha_i = 3,4$) spin components occur. Also, the superscript $^{(r)}$ means the coefficient of κ_p^r. For $v^0 > u^0$, $u^0 + 1/2 \leq p \leq v^0 - 1/2$, we obtain

$$\langle F(u)M(v)\rangle^{(3)} = -\frac{1}{6^2}\sum_{\vec{\gamma}\vec{g},\vec{w}}\left[\langle F(u)\bar{b}^{\ell}_{\vec{\gamma}\vec{g}}(p,\vec{w})\rangle^{(0)}\langle b^{\ell}_{\vec{\gamma}\vec{g}}(p+1,\vec{w})M(v)\rangle^{(0)} - \langle F(u)b^{u}_{\vec{\gamma}\vec{g}}(p,\vec{w})\rangle^{(0)}\langle \bar{b}^{u}_{\vec{\gamma}\vec{g}}(p+1,\vec{w})M(v)\rangle^{(0)}\right]\,.$$

For $v^0 < u^0$, $v^0 + 1/2 \leq p \leq u^0 - 1/2$, we obtain

$$\langle L(u)H(v)\rangle^{(3)} = \frac{1}{6^2}\sum_{\vec{\gamma}\vec{g},\vec{w}} \left[\langle L(u)b^{\ell}_{\vec{\gamma}\vec{g}}(p+1,\vec{w})\rangle^{(0)}\langle \bar{b}^{\ell}_{\vec{\gamma}\vec{g}}(p,\vec{w})H(v)\rangle^{(0)} - \langle L(u)\bar{b}^{u}_{\vec{\gamma}\vec{g}}(p+1,\vec{w})\rangle^{(0)}\langle b^{u}_{\vec{\gamma}\vec{g}}(p,\vec{w})H(v)\rangle^{(0)}\right], \quad (8)$$

or with the complex conjugate * everywhere.

For closure, meaning that the correlations on the lhs and rhs of Eq. (8) are the same, take $M = \bar{b}^{\ell}_{\vec{\beta}\vec{h}}$, $F = b^{\ell}_{\vec{\alpha}\vec{f}}$, then we get, for $u^0 < v^0$, $u^0 \leq p - 1/2 < v^0$,

$$\langle b^{\ell}_{\vec{\alpha}\vec{f}}(u)\bar{b}^{\ell}_{\vec{\beta}\vec{h}}(v)\rangle^{(3)} = -\frac{1}{6^2}\sum_{\vec{\gamma}\vec{g},\vec{w}} \langle b^{\ell}_{\vec{\alpha}\vec{f}}(u)\bar{b}^{\ell}_{\vec{\gamma}\vec{g}}(p,\vec{w})\rangle^{(0)}\,\langle b^{\ell}_{\vec{\gamma}\vec{g}}(p+1,\vec{w})\bar{b}^{\ell}_{\vec{\beta}\vec{h}}(v)\rangle^{(0)}\,. \quad (9)$$

For $u^0 > v^0$, $u^0 \geq p - 1/2 > v^0$, take $L = \bar{b}^{\ell}_{\vec{\alpha}\vec{f}}$, $H = b^{\ell}_{\vec{\beta}\vec{h}}$, then

$$\langle \bar{b}^{\ell}_{\vec{\alpha}\vec{f}}(u)b^{\ell}_{\vec{\beta}\vec{h}}(v)\rangle^{(3)} = \frac{1}{6^2}\sum_{\vec{\gamma}\vec{g},\vec{w}} \langle \bar{b}^{\ell}_{\vec{\alpha}\vec{f}}(u)b^{\ell}_{\vec{\gamma}\vec{g}}(p,\vec{w})\rangle^{(0)}\,\langle \bar{b}^{\ell}_{\vec{\gamma}\vec{g}}(p+1,\vec{w})b^{\ell}_{\vec{\beta}\vec{h}}(v)\rangle^{(0)}\,. \quad (10)$$

There are redundancies in the set of $\hat{b}_{\vec{\alpha}\vec{f}}$ fields and we want to eliminate linear dependencies such that we can associate baryon fields to particles (to have a basis). By doing this, we will be able to define an invertible two-point function. The $\hat{b}_{\vec{\alpha}\vec{f}}$ comprise a set of 216 fields, but by the t.s.p. there are equalities among them. We decompose the set into three classes as follows:

- Class 1 has all pairs $\alpha_i f_i$ the same in $\vec{\alpha}\vec{f}$, with 6 distinct elements;
- Class 2 has only two identical $\alpha_i f_i$, with 30 distinct elements;
- Class 3 has all distinct $\alpha_i f_i$, with 20 distinct elements.

Thus, altogether there are 56 distinct elements.

Taking into account multiplicities in the intermediate sums and restricting $\vec{\alpha}\vec{f}$ to only one representative of each class, using *tsp* and suppressing the lower spin component superscripts, we define normalized fields

$$\hat{B}_{\vec{\alpha}\vec{f}} = \frac{1}{n_{\vec{\alpha}\vec{f}}}\,\hat{b}_{\vec{\alpha}\vec{f}} \quad (11)$$

with $n_{\vec{\alpha}\vec{f}}$ equal 6, $2\sqrt{3}$ and $\sqrt{6}$ for the classes 1, 2 and 3, respectively. As we shall see, it turns out that the $\bar{B}$'s fields create baryons, and we call the basis of baryon fields of Eq. (11) the individual spin and flavor basis, or simply the *individual basis*, for short. From now on, we take into account the above restrictions without changing the notation.

Upon carrying out the sums of the product of two Levi-Civita's (ϵ's), for coincident points, and using the basic identity (for coincident points)

$$\langle b_{\vec{\alpha}\vec{f}}\bar{b}_{\vec{\beta}\vec{h}}\rangle^{(0)} = -6\,\mathrm{per}(\delta_{\vec{\alpha}\vec{\beta}}\,\delta_{\vec{f}\vec{h}})\,, \quad (12)$$

where we recall that, at $\kappa = 0$, our model also has a global SU(2) spin invariance in the space of lower and upper components. Then, for coincident points,

$$\langle B_{\vec{\alpha}\vec{f}}\bar{B}_{\vec{\beta}\vec{h}}\rangle^{(0)} = -\frac{6}{(n_{\vec{\alpha}\vec{f}})^2}\,\mathrm{per}(\delta_{\vec{\alpha}\vec{\beta}}\,\delta_{\vec{f}\vec{h}})\,,$$

where, here $^{(0)}$ means $\kappa = 0$ and, employing the ordinary Cayley notation for determinants, the 3×3 matrix $\delta_{\vec{\alpha}\vec{\beta}}\,\delta_{\vec{f}\vec{h}}$ has elements $\delta_{\alpha_i\beta_j}\,\delta_{f_i h_j}$ and where $\mathrm{per}(A)$ is the permanent of the square matrix A, which is defined like the determinant except that all terms have the plus sign. For the classes 1, 2 and 3 above, $\mathrm{per}(\delta_{\vec{\alpha}\vec{\alpha}}\,\delta_{\vec{f}\vec{f}})$ has the values 6, 2, 1, respectively, so that $\langle B_{\vec{\alpha}\vec{f}}\bar{B}_{\vec{\beta}\vec{h}}\rangle^{(0)} = -1$. Fixing a class, if $\vec{\alpha}\vec{f}$ is distinct from $\vec{\beta}\vec{h}$ then, $\langle B_{\vec{\alpha}\vec{f}}\bar{B}_{\vec{\beta}\vec{h}}\rangle^{(0)} = 0$, as can be seen from the lack of pairings. Thus, $\langle\,\cdot\,\bar{\cdot}\,\rangle^{(0)}$ provides us an inner product vector space generated by the individual basis vectors. (The above inner product is *not* to be confused with the physical Hilbert space inner product which is valid for all $\kappa > 0$.)

Define, using the $u^0 < v^0$ value to extend the correlation values to $u^0 = v^0$,

$$G_{\ell\ell'}(u,v) = \langle B_\ell(u)\bar{B}_{\ell'}\rangle\,\chi_{u^0\leq v^0} - \langle \bar{B}_\ell(u)B_{\ell'}\rangle^*\,\chi_{u^0>v^0}\,, \tag{13}$$

where the subscripts ℓ and ℓ' are now collective indices, and the lower spin subspace superscripts have been dropped. Note that, by time reversal, $\langle B_\ell(u)\bar{B}_{\ell'}(v)\rangle = -\langle \bar{B}_\ell(-u^0,\vec{u})B_{\ell'}(-v^0,\vec{v})\rangle^*$ so that

$$G_{\ell\ell'}(u,v) = G_{\ell\ell'}((-u^0,\vec{u}),(-v^0,\vec{v}))\,.$$

Remark 1 *As seen from the above discussion, we remark that the apparently awkward form of the correlation in Eq. (13) emerges naturally from the two time orderings in the Feynman-Kac formula.*

Before proceeding, we point out that there is another choice of F, M, L, H which also results in closure, namely, $L = b^u_{\vec{\alpha}\vec{f}}$, $M = b^u_{\vec{\beta}\vec{h}}$, $F = \bar{b}^u_{\vec{\alpha}\vec{f}}$, $H = \bar{b}^u_{\vec{\beta}\vec{h}}$. The associated B fields create antibaryons and are related to the baryon fields $\bar{B}$ by charge conjugation (see Ref. [35] and Appendix B). Baryons and antibaryons have identical spectral properties.

Returning to $G_{\ell_1\ell_2}$, we find, for $|u^0-v^0|>0$,

$$G^{(3)}_{\ell_1\ell_2}(u,v) = -\left\{\begin{array}{l} \sum_{\ell_3,\vec{w}}\, G^{(0)}_{\ell_1\ell_3}(u,(p,\vec{w}))\, G^{(0)}_{\ell_3\ell_2}((p+1,\vec{w}),v)\chi_{u^0<v^0}\,, \\ \sum_{\ell_3,\vec{w}}\, G^{(0)}_{\ell_1\ell_3}(u,(p+1,\vec{w}))\, G^{(0)}_{\ell_3\ell_2}((p,\vec{w}),v)\chi_{u^0>v^0}\,. \end{array}\right. \tag{14}$$

Write Eq. (14) symbolically as

$$G^{(3)} = -G^{(0)}\circ G^{(0)}\,, \tag{15}$$

which we refer to as the *product structure*. Note that the $\circ$ operation is a true convolution in space variables.

A spectral representation for G, for $u^0\neq v^0$, is obtained by using the spectral representations for the E-M operators in Eqs. (6) and (7). Take $L = \bar{B}_{\ell_1}$, $M = \bar{B}_{\ell_2}$, with $\bar{B}_\ell \equiv \bar{B}_\ell(1/2,\vec{0})$, letting $x \equiv u-v$, and recalling that $G(u,v) = G(u-v)$, it is given by

$$G_{\ell_1\ell_2}(x) \;=\; -\int_{-1}^{1}\int_{\mathbb{T}^3} (\lambda^0)^{|x^0|-1} e^{-i\vec{\lambda}.\vec{x}}\, d_\lambda(\bar{B}_{\ell_1},\mathcal{E}(\lambda^0,\vec{\lambda})\bar{B}_{\ell_2})_{\mathcal{H}}\,; \tag{16}$$

for $x\in\mathbb{Z}^4$, $x^0\neq 0$, and is an even function of $\vec{x}$ by parity symmetry.

Concerning the Fourier transform $\tilde{G}_{\ell_1\ell_2}(p) = \sum_{x\in\mathbb{Z}^4} G_{\ell_1\ell_2}(x)e^{-ip.x}$ of $G_{\ell_1\ell_2}(x)$, after separating the equal time contribution, it admits the spectral representation

$$\tilde{G}_{\ell_1\ell_2}(p) = \tilde{G}_{\ell_1\ell_2}(\vec{p}) - (2\pi)^3\int_{-1}^{1} f(p^0,\lambda^0)d_{\lambda^0}\alpha_{\vec{p},\ell_1\ell_2}(\lambda^0)\,, \tag{17}$$

where

$$d_{\lambda^0}\alpha_{\vec{p},\ell_1\ell_2}(\lambda^0) = \int_{\mathbb{T}^3} \delta(\vec{p}-\vec{\lambda})d_{\lambda^0}d_{\vec{\lambda}}(\bar{B}_{\ell_1},\mathcal{E}(\lambda^0,\vec{\lambda})\bar{B}_{\ell_2})_{\mathcal{H}}\,, \tag{18}$$

with $f(x,y)\equiv(e^{ix}-y)^{-1}+(e^{-ix}-y)^{-1}$, and we set $\tilde{G}(\vec{p}) = \sum_{\vec{x}} e^{-i\vec{p}.\vec{x}}G(x^0=0,\vec{x})$.

We use $\tilde{G}(p)$ to detect particles. Singularities of $\tilde{G}(p)$ are points in the E-M spectrum, and we take particles to correspond to singularities which form isolated curves $p^0 = iw(\vec{p})$.

First of all, we have a spectral mass gap as G has exponential decay which is one of the consequences of the polymer expansion method (see Refs. [29, 30]). A more precise bound on the decay is obtained using the hyperplane decoupling method and is given by:

Theorem 2 *The two-point function kernel $G_{\ell_1\ell_2}(u,v,\kappa)\equiv G_{\ell_1\ell_2}(u,v)\equiv G_{\ell_1\ell_2}(u-v)$ verifies the following global bound, with $|x| = |x^0| + \sum_{j=1}^3 |x^j|$,*

$$|G_{\ell_1\ell_2}(u,v)| \;<\leq \mathcal{O}(1)\,|\kappa|^{3|u-v|}\,, \tag{19}$$

for some positive constant $\mathcal{O}(1)$ uniform in κ and in the multindices ℓ_1 and ℓ_2. □

Remark 2 *From Theorem 2, we have a spectral mass gap of at least* $-(3-\epsilon)\ln\kappa$, $0<\epsilon\ll 1$.

To go higher in the spectrum, we make use of the meromorphic extension of $\tilde{G}(p)$ in p^0, for fixed $\vec{p}$ and κ, given by

$$\tilde{\Gamma}^{-1}(p) = \frac{\{\mathrm{cof}[\tilde{\Gamma}(p)]\}^t}{\det\tilde{\Gamma}(p)}\,, \tag{20}$$

where $\tilde{\Gamma}(p)\tilde{G}(p)=1$, such as Γ is the convolution inverse of G. Thus, the singularities of $\tilde{G}(p)$ are contained in the zeroes of $\det\tilde{\Gamma}(p)$ (see note [51]).

That $\tilde{\Gamma}^{-1}(p)$ provides an extension of $\tilde{G}(p)$ follows from the faster falloff of Γ, as compared to G. The product structure is instrumental in showing the faster temporal falloff of Γ. We define Γ by the Neumann series

$$\begin{aligned}\Gamma &= (1+G_d^{-1}G_n)^{-1}\,G_d^{-1} \\ &= \sum_{n=0}^{\infty}(-1)^n\,[G_d^{-1}G_n]^n\,G_d^{-1}\,,\end{aligned} \tag{21}$$

where we decompose G as $G=G_d+G_n$, with G_d diagonal, and G_d and G_d^{-1} bounded since $G_d^{(0)}=-1$. The series converges since $|G_n|$ is of order κ^3 and is small for small κ, as it follows from Theorem 2 . With this definition, we have:

Theorem 3 *The convolution inverse two-point function kernel* $\Gamma_{\ell_1\ell_2}(u,v,\kappa)\equiv\Gamma_{\ell_1\ell_2}(u,v)\equiv\Gamma_{\ell_1\ell_2}(u-v)$ *is bounded and satisfies*

$$|\Gamma_{\ell_1\ell_2}(u,v)| \;\le\; \mathcal{O}(1)\,|\kappa|^{3|\vec{u}-\vec{v}|}\,|\kappa|^{3+5(|u^0-v^0|-1)},\ |u^0-v^0|\neq 0\,, \tag{22}$$

for some positive constant $\mathcal{O}(1)$ *uniform in* κ *and in the multiple indices* ℓ_1 *and* ℓ_2*. The rhs is replaced by* const $\kappa^{3|\vec{u}-\vec{v}|}$, *if* $u^0=v^0$. □

Proof of Theorems 2 and 3: The proofs are direct applications of the hyperplane decoupling expansion as sketched above for G, and follow the proof of Theorem 1 of Ref. [35]. To compute the κ_p derivatives of Γ we differentiate the relation $\Gamma\,G=1=G\,\Gamma$, which gives the formula for κ_p derivatives at $\kappa_p=0$, $\partial^r\Gamma'=\sum_{s=0}^{r-1}\begin{pmatrix} r \\ s\end{pmatrix}\Gamma'\,\partial^{r-s}G\,\partial^s\Gamma'$ (with $\Gamma'\equiv-\Gamma$). We also need the product structure of Eq. (15), and the κ_p derivatives of G. The details will be omitted here. ■

From Theorem 3 , we see that $\tilde{\Gamma}(p)$ is analytic in the strip $|\mathcal{I}\mathrm{m}\,p^0|\le-(5-\epsilon)\ln\kappa$. Recalling that the dispersion curves are defined by the equation

$$\det\tilde{\Gamma}(p^0=iw(\vec{p}),\vec{p})=0\,, \tag{23}$$

it follows that, with fixed $\vec{p}$, the curves are isolated.

We still do not know the number and form of the solutions to Eq. (23). To obtain this information, we first give an intuitive argument. Retaining only terms to order κ^3, i.e. using only the values for distance zero and one given in Theorem 4 below, we have

$$\tilde{G}_{\ell_1\ell_2}(p)=[-1-2\kappa^3\cos p^0-\frac{\kappa^3}{4}\sum_{j=1,2,3}\cos p^j]\,\delta_{\ell_1\ell_2}+\mathcal{O}(\kappa^4)\,,$$

and

$$\tilde{\Gamma}_{\ell_1\ell_2}(p)=[-1+2\kappa^3\cos p^0+\frac{\kappa^3}{4}\sum_{j=1,2,3}\cos p^j]\,\delta_{\ell_1\ell_2}+\mathcal{O}(\kappa^4)\,.$$

Now, the important point to observe is that, dropping the $\mathcal{O}(\kappa^4)$ terms in $\tilde{\Gamma}(p)$, $\det\tilde{\Gamma}(p)$ factorizes into 56 identical factors. Under the above approximation, with

$$p_\ell^2\equiv 2\sum_{i=1}^{3}(1-\cos p^i)\,, \tag{24}$$

for each factor, we get identical dispersion curves of the form

$$w(\vec{p}) \equiv w(\vec{p},\kappa) = \left[-3\ln\kappa - \frac{3\kappa^3}{4} + \frac{\kappa^3}{8}p_\ell^2\right] + \mathcal{O}(\kappa^4)\,, \tag{25}$$

such that the particle mass is

$$M \equiv w(\vec{0},\kappa) = \left[-3\ln\kappa - \frac{3\kappa^3}{4}\right] + \mathcal{O}(\kappa^4)\,, \tag{26}$$

Note that the above solution in $\mathcal{I}\mathrm{m}\,p^0$ runs out to infinity as $\kappa \searrow 0$.

To find the solution without approximation, we pass to another basis of fields which is the particle basis introduced in the next subsection. In this basis, the matrix is more diagonal. This is seen by using the SU(3) flavor and other symmetries like time reversal, charge conjugation, parity and time reflection, and the analysis becomes much simpler.

It is important to observe that the particle basis is related to the individual basis by a real orthogonal transformation and that the product structure of the third hyperplane derivative is preserved. Thus, in the particle basis we also have faster decay of Γ as compared to G, which allows us to conclude that the dispersion curves are isolated from the rest of the spectrum.

C. Particle Basis, One-baryon Spectrum and the Eightfold Way

Here we introduce a new 56 dimensional basis, called the *particle basis* which is obtained from the individual basis by a real orthogonal transformation, and which is also orthonormal with the previous baryon inner product at $\kappa = 0$ given in Section IIb. In Appendix A, we show orthogonality relations and identities for the corresponding correlations; we also show how the elements of the particle basis are labelled by the eightfold way quantum numbers of third-component and total isospin, hypercharge and the values of the quadratic Casimir of $\mathrm{SU}(3)_f$, as well as the labels of total spin and its z-component. (Note that the cubic Casimir is not needed here!)

For the particle basis, the associated two-point function matrix is diagonal in all of these quantum numbers, except for the spin, for all $\kappa > 0$. Using $\mathrm{SU}(3)_f$, it decomposes into eight *identical* 2×2 blocks associated with total spin 1/2 octet baryons and ten *identical* 4×4 blocks, associated with total spin 3/2 decuplet. The convolution inverse Γ has the same block structure.

The use of $\mathrm{SU}(3)_f$ symmetry alone still does not allow us to conclude the diagonality of the 2×2 and 4×4 blocks. Using a *new symmetry*, the local spin flip symmetry $\mathcal{F}_s$ given in Section IV (see also Appendix B), together with parity $\mathcal{P}$, and time reversal $\mathcal{T}$ allows us to conclude the 2×2 block is diagonal and a multiple of the identity. The effect of applying this symmetry to the 4×4 block is to produce some zeroes and relations between the elements.

The structure of $\Gamma(x)$ given above carries over to $\tilde{\Gamma}(p^0 = i\chi, \vec{p})$, χ real. For the determination of masses, $\vec{p} = \vec{0}$, the use of a $\pi/2$ rotation about the e^3 axis [see Eq. (A15)] shows that the 4×4 block of $\tilde{\Gamma}(\vec{p} = \vec{0})$ is diagonal, and the use of reflection symmetry in e^1 shows that the diagonal elements are pairwise equal. Hence, at $\vec{p} = \vec{0}$, $\det\tilde{\Gamma}$ factorizes into 56 factors and the masses can be determined using the auxiliary function method, which we do at the end of this section (see also Ref. [35]). The particle masses are given by convergent expansions in κ.

The block structure of $\tilde{\Gamma}(\vec{p})$ is also exploited in the determination of dispersion *curves*. The fact that the 4×4 blocks are not diagonal puts some limitation in our analysis of dispersions for the decuplet, as explained below.

The decoupling of hyperplane method shows that the local, gauge-invariant fields which create baryons are given in the two next equations (with all fields evaluated at the same point). For the anti-baryons, *all* the fields must be unbarred, and only upper spin components occur. Concerning the octet, we define the following baryon particle fields:

$$\begin{cases} p_\pm = \frac{\epsilon_{abc}}{3\sqrt{2}}\,(\bar\psi_{a+u}\bar\psi_{b-d} - \bar\psi_{a+d}\bar\psi_{b-u})\bar\psi_{c\pm u}, \\ n_\pm = \frac{\epsilon_{abc}}{3\sqrt{2}}\,(\bar\psi_{a+u}\bar\psi_{b-d} - \bar\psi_{a+d}\bar\psi_{b-u})\bar\psi_{c\pm d}, \\ \Xi^0_\pm = \frac{\epsilon_{abc}}{3\sqrt{2}}\,(\bar\psi_{a+u}\bar\psi_{b-s} - \bar\psi_{a+s}\bar\psi_{b-u})\bar\psi_{c\pm s}, \\ \Xi^-_\pm = \frac{\epsilon_{abc}}{3\sqrt{2}}\,(\bar\psi_{a+d}\bar\psi_{b-s} - \bar\psi_{a+s}\bar\psi_{b-d})\bar\psi_{c\pm s}, \\ \Sigma^+_\pm = \frac{\epsilon_{abc}}{3\sqrt{2}}\,(\bar\psi_{a+u}\bar\psi_{b-s} - \bar\psi_{a+s}\bar\psi_{b-u})\bar\psi_{c\pm u}, \\ \Sigma^0_\pm = \frac{\epsilon_{abc}}{6}\,(2\bar\psi_{a\pm u}\bar\psi_{b\pm d}\bar\psi_{c\mp s} - \bar\psi_{a-u}\bar\psi_{b+d}\bar\psi_{c\pm s} - \bar\psi_{a+u}\bar\psi_{b-d}\bar\psi_{c\pm s}), \\ \Sigma^-_\pm = \frac{\epsilon_{abc}}{3\sqrt{2}}\,(\bar\psi_{a+d}\bar\psi_{b-s} - \bar\psi_{a+s}\bar\psi_{b-d})\bar\psi_{c\pm d}, \\ \Lambda_\pm = \frac{\epsilon_{abc}}{2\sqrt{3}}\,(\bar\psi_{a+u}\bar\psi_{b-d} - \bar\psi_{a+d}\bar\psi_{b-u})\bar\psi_{c\pm s}\,, \end{cases} \tag{27}$$

Here, the subindexes $\pm$ denote the z-component of total spin $J_z = \pm\frac{1}{2}$. The particle quantum numbers are given in Figure 1 below. n, p, Ξ^- and Ξ^0 have total isospin $I = 1/2$; Σ^+, Σ^0 and Σ^- have $I = 1$ and Λ has $I = 0$. The associated physical Hilbert space states are shown to have the isospin, hypercharge and strangeness quantum numbers displayed in Figure 1. The value of the quadratic Casimir C_2 is equal to 3 for each member of the octet. In Eq. (27), the particle symbol is on the lhs and the barred field that creates it in on the rhs.

Remark 3 *The following observations concerning Figure 1 are important:*

1. *For p, n the first two isospins are coupled to give isospin zero. Σ^0 is obtained from Σ^+ by applying the isospin lowering operator.*
2. *For each term of $\Sigma^\pm$, $I_3 = \pm 1$, $I_\pm \Sigma^\pm = 0$, $I_\pm I_\mp \Sigma^\pm = 2\Sigma^\pm$ and, as $\vec{I}^2 = (I_+ I_- + I_- I_+)/2 + I_3^2$, $\vec{I}^2 \Sigma^\pm = 2\Sigma^\pm$, such as $I = 1$. The same holds for Σ^0.*
3. *$\Lambda_\pm$ is antisymmetric in $u \leftrightarrow d$ so that $I = 0$.*
4. *For all but Σ^0, the first two spins are coupled to give spin zero, i.e. antisymmetric in the first two spins, as seen using tsp in $\alpha_1 f_1 \leftrightarrow \alpha_2 f_2$.*
5. *For each fixed spin J_z, these fields form a basis for the eight dimensional representation of $\mathrm{SU}(3)_f$.*

For the decuplet, we define:

$$
\begin{cases}
\Delta^+_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{6}(\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\mp d} + 2\bar\psi_{a\pm u}\bar\psi_{b\mp u}\bar\psi_{c\pm d}), \\
\Delta^+_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\,\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\pm d}, \\
\Delta^0_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{6}(2\bar\psi_{a\pm u}\bar\psi_{b\pm d}\bar\psi_{c\mp d} + \bar\psi_{a\mp u}\bar\psi_{b\pm d}\bar\psi_{c\pm d}), \\
\Delta^0_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\,\bar\psi_{a\pm u}\bar\psi_{b\pm d}\bar\psi_{c\pm d}, \\
\Delta^-_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\,\bar\psi_{a\pm d}\bar\psi_{b\pm d}\bar\psi_{c\mp d}\,, \\
\Delta^-_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{6}\,\bar\psi_{a\pm d}\bar\psi_{b\pm d}\bar\psi_{c\pm d}\,, \\
\Delta^{++}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\,\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\mp u}\,, \\
\Delta^{++}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{6}\,\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\pm u}\,, \\
\Sigma^{*+}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\pm s}, \\
\Sigma^{*+}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{6}\,(\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\mp s} + 2\bar\psi_{a\pm u}\bar\psi_{b\mp u}\bar\psi_{c\pm s})\,, \\
\Sigma^{*0}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{6}\,\bar\psi_{a\pm u}\bar\psi_{b\pm d}\bar\psi_{c\pm s}\,, \\
\Sigma^{*0}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{3\sqrt{2}}(\bar\psi_{a\pm u}\bar\psi_{b\pm d}\bar\psi_{c\mp s} + \bar\psi_{a\pm u}\bar\psi_{b\mp d}\bar\psi_{c\pm s} + \bar\psi_{a\mp u}\bar\psi_{b\pm d}\bar\psi_{c\pm s}), \\
\Sigma^{*-}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar\psi_{a\pm d}\bar\psi_{b\pm d}\bar\psi_{c\pm s}, \\
\Sigma^{*-}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{6}\,(\bar\psi_{a\pm d}\bar\psi_{b\pm d}\bar\psi_{c\mp s} + 2\bar\psi_{a\pm d}\bar\psi_{b\mp d}\bar\psi_{c\pm s}), \\
\Xi^{*0}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar\psi_{a\pm u}\bar\psi_{b\pm s}\bar\psi_{c\pm s}, \\
\Xi^{*0}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{6}\,(\bar\psi_{a\mp u}\bar\psi_{b\pm s} + 2\bar\psi_{a\pm u}\bar\psi_{b\mp s})\bar\psi_{c\pm s}, \\
\Xi^{*-}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar\psi_{a\pm d}\bar\psi_{b\pm s}\bar\psi_{c\pm s}, \\
\Xi^{*-}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{6}\,(\bar\psi_{a\mp d}\bar\psi_{b\pm s} + 2\bar\psi_{a\pm d}\bar\psi_{b\mp s})\bar\psi_{c\pm s}, \\
\Omega^-_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{6}\bar\psi_{a\pm s}\bar\psi_{b\pm s}\bar\psi_{c\pm s}, \\
\Omega^-_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar\psi_{a\pm s}\bar\psi_{b\pm s}\bar\psi_{c\mp s}\,,
\end{cases}
\tag{28}
$$

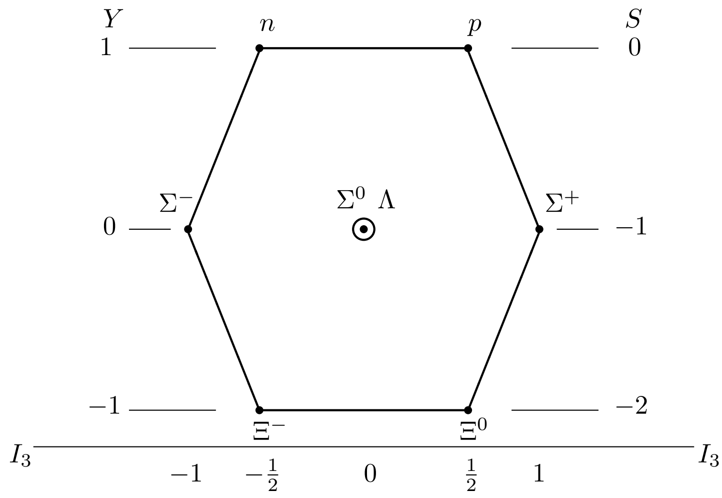

Figure 1: The Baryon Octet. We use the same particle symbols for the corresponding Grassmannian fields. The particle total hypercharge Y, strangeness S, total isospin I and third component of total isospin I_3 are indicated.

The fields given in Eq. (28) are associated with the baryon decuplet particles and are represented in Figure 2. The value of total isospin I equals 3/2 for Δ^-, Δ^0, Δ^+ and Δ^{++}; $I = 1$ for Σ^{*-}, Σ^{*+} and Σ^{*0}; $I = 1/2$ for Ξ^{*-} and Ξ^{*0}; and $I = 0$ for Ω^-. The value of the quadratic Casimir C_2 is equal to 6 for each member of the decuplet.

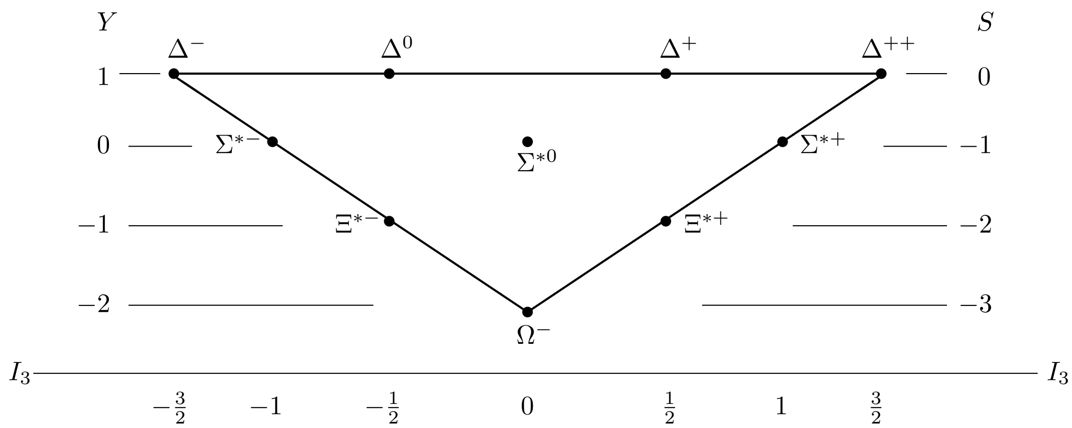

Figure 2: The Baryon Decuplet.

Remark 4 *The following observations concerning Figure 2 are important:*

1. Σ^{*0} *is obtained from* Σ^{*+} *by applying the isospin lowering operator. The spin* $J_z = 1/2$ *states are obtained from the spin* $J_z = 3/2$ *states by applying the spin lowering operator. The spin* $J_z = -1/2$ *states are obtained from the spin* $J_z = -3/2$ *states by applying the spin raising operator.*

2. *For each fixed spin* J_z*, these fields form a basis for the ten dimensional representation of* $\mathrm{SU}(3)_f$.

3. *Note that, modulo a correct choice of the spin matrices* γ^μ*, the form of the fields we obtain for the decuplets and the octets is* the same *as appears in continuum spacetime QCD.*

Except for the proton, neutron and Λ, which have electric charges $+1$, 0 and 0, respectively, all other total charges Q are indicated as upper indices. The particle electric charges verify the Gell'Mann-Nishijima relation

$$Q \,=\, I_3 \,+\, \frac{Y}{2}\,,$$

From Eqs. (27) and (28) we see that the particle fields are obtained from the individual basis fields by a real linear transformation. Since, for coincident points,

$$\langle\, B_{r_1}\, \bar{B}_{r_2}\,\rangle^{(0)} \;=\; -\delta_{r_1 r_2}\,,$$

where r_1, r_2 are collective indices, we see that this transformation is orthogonal.

With an abuse of notation, we denote the baryon-baryon correlation function in the particle basis by $G_{r_1 r_2}(u,v)$, reserving the r indices for the particle basis. In Appendix A, using $\mathrm{SU}(3)_f$ symmetry as well as $\mathcal{PCT}$ and other symmetries, we establish important correlation function orthogonality relations and identities which are used to reduce the matrix $G_{r_1 r_2}(x)$. As the individual and the particle basis are related by a real orthogonal transformation, it is seen that the product structure of Eq. (14) is preserved for the third hyperplane κ_q derivative of $G_{r_1 r_2}$ at $\kappa_q = 0$. As before, the product structure is a true convolution is space variables. Consequently, for the spatial Fourier transform, we have

$$\hat{G}^{(3)}_{r_1 r_2}(u^0,v^0;\vec{p}) = \begin{cases} -\sum_r\, \hat{G}^{(0)}_{r_1 r}(u^0,q;\vec{p})\, \hat{G}^{(0)}_{r r_2}(q+1,v^0;\vec{p})\,,\; u^0 < v^0\,, \\ -\sum_r\, \hat{G}^{(0)}_{r_1 r}(u^0,q+1;\vec{p})\, \hat{G}^{(0)}_{r r_2}(q,v^0;\vec{p})\,,\; u^0 > v^0\,. \end{cases}$$

Again, this product structure implies the faster falloff of $\Gamma_{r_1 r_2}(x)$ of Theorem 3, and the larger analyticity region for $\tilde{\Gamma}_{r_1 r_2}(p)$. However, $\tilde{\Gamma}_{r_1 r_2}$ is still not yet seen to be diagonal due to the non-diagonality in spin at $\kappa \neq 0$. As explained in the beginning of this section, for $\vec{p} = \vec{0}$, it is diagonal. So, for each one of the 56 r values, we can use the auxiliary function method to obtain the particle mass (see below). Each particle mass has the form

$$M = -3\ln\kappa + r(\kappa)\,,$$

where $r(\kappa)$ is real analytic in κ. The $r_(\kappa)$'s, and then the masses, are all equal for the particles in the octet. For the decuplet, they are also equal up to and including order κ^6. The octet masses may differ from the ones of the decuplet. The low order terms in the κ expansion for all the masses are $M = -3\ln\kappa - 3\kappa^3/4 + c\kappa^6 + \mathcal{O}(\kappa^7)$, where c is a constant which has different values for the octet/decuplet. Up to and including order κ^3, this result agrees with the expression obtained using the individual basis [see Eq. (26)]. So, there is mass splitting between particles in the octet and those in the decuplet at order κ^6. In fact, the coefficient of $\kappa^{(n)}$ of $r(\kappa)$ is determined by considering only a finite-volume correlation, with linear dimension of roughly $n/3$.

In order to make a finer analysis of the masses, we now go back to the dispersion curve which are solutions of Eq. (23). To find the solution without approximation, we make a nonlinear transformation from p^0 to an auxiliary variable w and introduce an auxiliary matrix function $H_{\ell\ell'}(w,\kappa)$ to bring the solution for the nonsingular part $w(\vec{p}) + 3\ln\kappa$ of the dispersion curves from infinity to close to $w = 0$ for small κ. With this function we can cast the problem of determining dispersion curves and masses into the framework of the analytic implicit function theorem.

To this end, we let

$$w = -1 - c_3(\vec{p})\kappa^3 + \kappa^3 e^{-ip^0} \tag{29}$$

with $c_3(\vec{p}) = -\sum_{j=1,2,3}\cos p^j/4$, so that

$$\tilde{\Gamma}(p^0,\vec{p}) = H(w = -1 - c_3(\vec{p})\kappa^3 + \kappa^3 e^{-ip^0},\kappa)\,,$$

where the auxiliary function $H(w,\kappa)$ is defined by

$$\begin{aligned} H(w,\kappa) \;=\; & \sum_{\vec{x}} \Gamma(x^0=0,\vec{x})\, e^{-i\vec{p}.\vec{x}} + \\ & \sum_{\vec{x},n=1,2\ldots} \Gamma(n,\vec{x})\, e^{-i\vec{p}.\vec{x}} \left[\left(\frac{1+w+c_3(\vec{p})\kappa^3}{\kappa^3}\right)^n + \left(\frac{\kappa^3}{1+w+c_3(\vec{p})\kappa^3}\right)^n\right]. \end{aligned} \tag{30}$$

It turns out that $H(w,\kappa)$ is jointly analytic in w, κ and p^j, for $|w|$, $|\kappa|$ and $|\mathcal{I}\mathrm{m}\, p^j|$ small. This is not apparent from the above due to the terms with a factor κ^{-3n}. These terms are regular by the faster decay for $\Gamma(x)$ of Theorem 3 , that we have obtained using the hyperplane decoupling expansion, except for some small distance points. For these points there are cancellations in the Neumann series of Eq. (21) defining Γ, which shows that these terms are regular. Also, the faster decay of Γ ensures convergence of the series given in Eq. (30).

For determining the masses, using symmetries as discussed in Section I, $\tilde\Gamma(p^0,\vec p=\vec 0)$ is shown to be diagonal and the determinant in Eq. (23) factorizes. The masses are determined by setting each diagonal element to zero. Without changing notation, we call H above one of these factors. Then, the implicit equation $H(w,\kappa)=0$ admits an analytic solution $w(\kappa)$. Using Eq. (29) and putting $p^0=iM$, the mass is given by $M=-3\ln\kappa+\ln[1+c_3(\vec 0)\kappa^3+w(\kappa)]$.

Remark 5 *To obtain the masses up to and including order κ^4, we need to isolate all κ^r, $r=0,1,2,3,4$ terms in $H(w,\kappa)$. To do this, we only need to consider the points $|x^0|=1$, $|\vec x|\leq 1$; $|x^0|=2$, $\vec x=\vec 0$; $x^0=0$, $|\vec x|\leq 1$; and show that the κ^6 contribution to Γ is zero for $|x^0|=1$, $|\vec x|=1$; and that the κ^8 and κ^9 contributions are zero for $|x^0|=2$, $\vec x=\vec 0$. The hyperplane decoupling method global bounds of Theorems 2 and 3 are sufficient to control Γ for all other points.*

To further our knowledge of M up to and including order κ^6, we must compute the values of the κ^{3r+6} contributions to $\Gamma(x^0=r\epsilon e^0,\vec x)$ for $r=0$, $|\vec x|\leq 2$; $r=1$, $|\vec x|\leq 2$; $r=2$, $|\vec x|\leq 1$; $r=3,4$, $\vec x=\vec 0$. As Γ is determined by the Neumann expansion of Eq. (21), we also need to know G for the above points and also the κ^{3r+3} contributions to $G(x)$ for $x=r\epsilon e^0+\epsilon' e^j$, $r=3,4$ and ϵ, $\epsilon'=\pm1$.

In obtaining these results we first need to know how to compute some gauge field correlations. These gauge correlations are derived using a generating function integral representation method and are given in Eqs. (B1)-(B4). Regarding the short distance behavior of $G(x)$ and $\Gamma(x)$ in the particle basis, we obtain:

Theorem 4 *Let $0<\kappa\ll 1$, $\mu,\nu=0,1,\ldots,d=3$ and e^0, e^i and e^j, $i,j=1,\ldots,d=3$, denote the time and space unit vectors, and $\epsilon,\epsilon',\epsilon''=\pm1$. The short-distance behaviors appearing below hold for G and Γ. We start with the decuplet two-point functions. In this case, we have*

$$G_{r_1r_2}(x)=\begin{cases}[-1+c_8\kappa^8+\mathcal{O}(\kappa^9)]\delta_{r_1r_2} & ,\quad x=0\,;\\ (-\kappa^3+c_9\kappa^9)\delta_{r_1r_2}+\mathcal{O}(\kappa^{11}) & ,\quad x=\epsilon e^0\,;\\ -\frac18\kappa^3\delta_{r_1r_2}+\mathcal{O}(\kappa^9) & ,\quad x=\epsilon e^j\,;\\ \left[(-\kappa^6+c_{12}\kappa^{12})\delta_{r_1r_2}+\mathcal{O}(\kappa^{13})\right]\delta_{\mu\,0}+\left[-\frac18\kappa^6\delta_{r_1r_2}+\mathcal{O}(\kappa^{10})\right]\delta_{\mu\,j} & ,\quad x=2\epsilon e^\mu\,;\\ (\frac1{16}\delta_{r_1\frac32}-\frac1{16}\delta_{r_1\frac12})\kappa^6\delta_{r_1r_2}+\mathcal{O}(\kappa^{10}) & ,\quad x=\epsilon e^1+\epsilon' e^2\,;\\ (-\frac1{32}\delta_{r_1\frac32}+\frac1{32}\delta_{r_1\frac12})\kappa^6\delta_{r_1r_2}+\mathcal{O}(\kappa^{10}) & ,\quad x=\epsilon e^1+\epsilon' e^3,\ \epsilon e^2+\epsilon' e^3\,;\\ -\frac14\kappa^6\delta_{r_1r_2}+\mathcal{O}(\kappa^{10}) & ,\quad x=\epsilon e^0+\epsilon' e^j\,;\\ \frac{17}{64}\kappa^9\delta_{r_1r_2}+\mathcal{O}(\kappa^{10}) & ,\quad x=\epsilon e^0+2\epsilon' e^j\,;\\ -\frac38\kappa^9\delta_{r_1r_2}+\mathcal{O}(\kappa^{13}) & ,\quad x=2\epsilon e^0+\epsilon' e^j\,;\\ (\frac3{32}\delta_{s\frac32}-\frac5{32}\delta_{s\frac12})\kappa^9\delta_{r_1r_2}+\mathcal{O}(\kappa^{10}) & ,\quad x=\epsilon e^0+\epsilon' e^1+\epsilon'' e^2\,;\\ (-\frac3{32}\delta_{s\frac32}+\frac1{32}\delta_{s\frac12})\kappa^9\delta_{r_1r_2}+\mathcal{O}(\kappa^{10}) & ,\quad x=\epsilon e^0+\epsilon' e^i+\epsilon'' e^3,\ i=1,2\,;\\ (-\kappa^9+c_{15}\kappa^{15})\delta_{r_1r_2}+\mathcal{O}(\kappa^{16}) & ,\quad x=3\epsilon e^0\,;\\ d_{12}\kappa^{12}\delta_{r_1r_2}+\mathcal{O}(\kappa^{13}) & ,\quad x=3\epsilon e^0+\epsilon' e^j\,;\\ (-\kappa^{12}+c_{18}\kappa^{18})\delta_{r_1r_2}+\mathcal{O}(\kappa^{19}) & ,\quad x=4\epsilon e^0\,;\\ d_{15}\kappa^{15}\delta_{r_1r_2}+\mathcal{O}(\kappa^{16}) & ,\quad x=4\epsilon e^0+\epsilon' e^j\,;\end{cases}\tag{31}$$

and

$$
\Gamma_{r_1 r_2}(x) \;=\; \begin{cases}
[-1 - \frac{67}{32}\kappa^6 + \mathcal{O}(\kappa^8)]\delta_{r_1 r_2} & , \quad x = 0\,; \\
(\kappa^3 + c_9'\kappa^9)\delta_{r_1 r_2} + \mathcal{O}(\kappa^{10}) & , \quad x = \epsilon e^0\,; \\
\frac{1}{8}\kappa^3\delta_{r_1 r_2} + \mathcal{O}(\kappa^9) & , \quad x = \epsilon e^j\,; \\
\left[c_{12}'\kappa^{12}\delta_{r_1 r_2} + \mathcal{O}(\kappa^{13})\right]\delta_{\mu\, 0} + \left[\frac{7}{64}\kappa^6\delta_{r_1 r_2} + \mathcal{O}(\kappa^{10})\right]\delta_{\mu\, j} & , \quad x = 2\epsilon e^\mu\,; \\
(-\frac{3}{32}\delta_{r_1 \frac{3}{2}} + \frac{1}{32}\delta_{r_1 \frac{1}{2}})\kappa^6\delta_{r_1 r_2} + \mathcal{O}(\kappa^{10}) & , \quad x = \epsilon e^1 + \epsilon' e^2\,; \\
(0\,\delta_{r_1 \frac{3}{2}} - \frac{1}{16}\delta_{r_1 \frac{1}{2}})\kappa^6\delta_{r_1 r_2} + \mathcal{O}(\kappa^{10}) & , \quad x = \epsilon e^1 + \epsilon' e^3,\ \epsilon e^2 + \epsilon' e^3\,; \\
\mathcal{O}(\kappa^{10}) & , \quad x = \epsilon e^0 + \epsilon' e^j\,; \\
\mathcal{O}(\kappa^{10}) & , \quad x = \epsilon e^0 + 2\epsilon' e^j\,; \\
\mathcal{O}(\kappa^{13}) & , \quad x = 2\epsilon e^0 + \epsilon' e^j\,; \\
\mathcal{O}(\kappa^{10}) & , \quad x = \epsilon e^0 + \epsilon' e^i + \epsilon'' e^{j>i}\,; \\
c_{15}'\kappa^{15}\delta_{r_1 r_2} + \mathcal{O}(\kappa^{16}) & , \quad x = 3\epsilon e^0\,; \\
c_{18}'\kappa^{18}\delta_{r_1 r_2} + \mathcal{O}(\kappa^{19}) & , \quad x = 4\epsilon e^0\,;
\end{cases} \tag{32}
$$

For the octet two-point function, the short distance behaviors for $x = r\epsilon e^\mu$, $r = 1,2,3,4$, $\epsilon e^0 + \epsilon' e^j$, $\epsilon e^0 + 2\epsilon' e^j$, $2\epsilon e^0 + \epsilon' e^j$, $3\epsilon e^0 + \epsilon' e^j$, $4\epsilon e^0 + \epsilon' e^j$ are the same as for the decuplet, up to the considered order in each case. However, for $x = \epsilon e^i + \epsilon' e^j$, $x = \epsilon e^0 + \epsilon' e^i + \epsilon'' e^j$, for $ij = 12, 13, 23$, the results are different and given respectively by

$$
G_{r_1 r_2} = -\frac{1}{16}\kappa^6\delta_{r_1 r_2} + \mathcal{O}(\kappa^7) \quad , \quad \Gamma_{r_1 r_2} = \frac{1}{32}\kappa^6\delta_{r_1 r_2} + \mathcal{O}(\kappa^7)\,, \tag{33}
$$

and

$$
G_{r_1 r_2} = -\frac{5}{32}\kappa^9\delta_{r_1 r_2} + \mathcal{O}(\kappa^{10}) \quad , \quad \Gamma_{r_1 r_2} = \mathcal{O}(\kappa^{10})\,. \tag{34}
$$

In the above, the c's, c''s, d's and d''s are computable κ and spin independent constants. □

Proof of Theorem 4 : The proof is given in Appendix B. ■

Remark 6 *We note that the absence of lower order terms in $\Gamma_{r_1 r_2}$, as compared to $G_{r_1 r_2}$, for $|x^0| = 2,3,4$, as well as for the points $x = \epsilon e^0 + \epsilon' e^j$, $\epsilon e^0 + 2\epsilon' e^j$ and $\epsilon e^0 + \epsilon' e^i + \epsilon'' e^{j>i}$, is due to explicit cancellations in the Neumann series and improves the hyperplane method bounds obtained in Theorem 2 .*

Remark 7 *An important observation concerning mass splitting is the fact that although the pointwise values for the spatial* angle *contributions $\Gamma(x = \epsilon e^i + \epsilon' e^j)$ in the decuplet differ for spin $1/2$ and $3/2$, the sum over the spatial points in Eq. (30) gives the same value. So, there is no mass splitting inside the decuplet up to and including order κ^6. However the sum value for the octet is different than that of the decuplet which gives rise to the decuplet/octet mass splitting given in Theorem* 1*.*

Remark 8 *Similar results for two-baryon functions, as the ones in Theorem 4, apply to the $3+1$ dimensional one and two-flavor case treated in Refs. [34, 35, 41, 42]. The results for Γ presented in Theorem 4 are in contrast to the one-flavor case. There we find no mass splitting between the baryon spin $1/2$ and spin $3/2$ states. However, a correction for the values for the angle contributions $x = \epsilon e^\mu + \epsilon' e^\nu$ for Γ, as above, shows that a mass splitting also occurs at order κ^6, between the proton-neutron and the delta particles in the $3+1$-dimensional, two-flavor case. This is not what is stated in Refs. [41, 42]. We emphasize that the main result in these papers, regarding the existence of two-baryon bound states in the spectrum is* not *affected by this since, for the bound state analysis using a lattice B-S equation, it is enough to know the baryon masses up to an order below κ^6, as the binding energies are of order $\mathcal{O}(\kappa^2)$.*

Continuing, we return to the determination of dispersion curves. Fixing κ and $\vec{p}$, we now apply a Rouché's theorem argument (see Ref. [30, 56]) to the analytic function

$$
\begin{aligned}
f(w) \equiv \det H(w,\kappa) \;&=\; \det w I_{56} + [\det H - \det w I_{56}] \\
&\equiv\; g(w) + h(w)\,.
\end{aligned}
$$

In the disc $|w| \le c\kappa^6$, $c \gg 1$, $|g(w)| > |h(w)|$, so that the equations $f(w) = 0$ and $g(w) = 0$ have the same number of solutions. Since $g(w) = w^{56} = 0$ has 56 solutions, there are precisely 56 not necessarily distinct solutions for the dispersion curves.

Going back to the relation between p^0 and w, and noting that $\ln(1 + c_3(\vec{p})\kappa^3 + w) = c_3(\vec{p})\kappa^3 + \mathcal{O}(\kappa^6)$, we see that each dispersion curve has the form given in Eq. (25). That each dispersion curve is isolated in the E-M spectrum in all $\mathcal{H}$, up to near the meson-baryon threshold of $-5\ln\kappa$, is shown in Section IV.

Remark 9 *We point out that the solutions $w(\vec{p})$ of Eq. (23), and then the dispersion curves, do* not *depend on the orthonormal basis we use.*

Remark 10 *Rouché's theorem can be applied without factorization in* $\det\Gamma$. *However,* not *necessarily will the solution to Eq. (23) define curves, but simply a collection of points.*

Now, recalling the block decomposition of $\tilde{\Gamma}$ described above, for $\vec{p} \neq \vec{0}$, for each of the 2×2 blocks of the 56×56 two-point function matrix formed with the $J_3 = \pm 1/2$ states of the same particle (with fixed values for I, I_z, Y and for the quadratic Casimir C_2) is diagonal, and all diagonal elements are equal. We can also apply here the auxiliary function method to find the octet dispersion curve $w_o(\vec{p})$. It is given by, recalling that $p_\ell^2 \equiv 2\sum_{i=1}^3 (1 - \cos p^i)$ as defined in Eq. (24),

$$w_o(\vec{p}) \equiv w(\vec{p},\kappa) = \left[-3\ln\kappa - 3\kappa^3/4 + p_\ell^2\kappa^3/8\right] + r_o(\kappa,\vec{p})\,, \tag{35}$$

where $r_o(\kappa,\vec{p})$ is of $\mathcal{O}(\kappa^6)$ and is jointly analytic in each component of $\vec{p}$, for small $|\mathcal{I}\mathrm{m}\, p^j|$, and κ.

For the decuplet, there are 10 identical 4×4 blocks and for $\vec{p} \neq \vec{0}$ each block is not diagonal but has two multiplicity two eigenvalues. In terms of the 4×4 matrix elements, the determining equations for two-by-two identical dispersion curves are [see Eq. (A14)]

$$\frac{1}{2}\left[\tilde{\Gamma}_{11} + \tilde{\Gamma}_{33}\right] \pm \sqrt{\frac{1}{4}\left[\tilde{\Gamma}_{11} - \tilde{\Gamma}_{33}\right]^2 + \left|\tilde{\Gamma}_{13}\right|^2 + \left|\tilde{\Gamma}_{14}\right|^2} = 0\,,$$

where the indices $1,2,3,4$ correspond to $J_z = 3/2, -3/2, 1/2, -1/2$, respectively.

Due to the square root above and our lack of knowledge of the order in κ of the terms under the square root, the auxiliary function method is not directly applicable. However, Rouché's argument can be applied and shows the existence of exactly four pairwise equal solutions. However, taking the union of these solutions for all $\vec{p}$ it does not tell us how to decompose them into curves.

Also, a fixed point method is sketched in Ref. [42] for constructing the dispersion curves.

We now concentrate on $\vec{p} = \vec{0}$, and show how to obtain convergent expansions for the masses and also the octet-decuplet mass splitting at order κ^6.

Using the global bounds of Theorem 3 and the short distance behavior of $\Gamma(x)$ given in Theorem 4 , we explicitly separate out the contributions to $H(w,\kappa)$ in Eq. (30) up to and including order κ^6. To this end, we can write $H(w,\kappa)$ for

$$n = 0\ :\quad = -1 - c_3(\vec{p})\kappa^3 + a_6\kappa^6 + b_6\kappa^6 + \kappa^7 r_0(\kappa)\,,$$

where we have explicitly separated the $a_6\kappa^6$ contribution from all spatial angles, i.e. contributions to $\Gamma(x)$ with $x = \epsilon e^i + \epsilon' e^j$;

$$n = 1\ :\quad = 1 + w + c_3(\vec{p})\kappa^3 + \frac{\kappa^6}{1+w} + b_1\kappa^6 + c_9'(1+w)\kappa^6 + \kappa^7 r_1(\kappa,w)\,,$$

and

$$n \ge 2\ :\quad = b_2\kappa^6 + \kappa^6 \sum_{k=2,\ldots,4} c_{3k+6}'(1+w)^k + \kappa^7 r_2(\kappa,w)\,.$$

Above, b_0, b_1, b_2 and the c''s are constants independent of the isospin (octet and decuplet). On the other hand, a_6 is a constant separately for the octet and the decuplet, and takes different values for each of them. Namely, from Theorem 4 , we have

$$a_6 \equiv a_o = \frac{3}{8}\ (octet)\quad ,\quad a_6 \equiv a_d = -\frac{3}{8}\ (decuplet)\,.$$

Moreover, $r_{0,1,2}$ are jointly analytic in w and κ.

Taking into account the above, we can write $H(w,\kappa)$ as

$$H(w,\kappa) \;=\; w + \frac{\kappa^6}{1+w} + a_6\kappa^6 + b\kappa^6 + \kappa^6 \sum_{k=1}^{4} c'_{3k+6}(1+w)^k + h(w,\kappa)\kappa^7 \,,$$

with b taking the same value for the octet and the decuplet, and $h(w,\kappa)$ jointly analytic in w and κ. As $H(0,0)=0$ and $[\partial H \partial w](0,0)=1$, the analytic implicit function theorem implies that the equation $H(w,\kappa)=0$ has the analytic solution $w(\kappa)$ given by, with $b' = b + 1 + \sum_{n=1,\ldots,4} c'_{3n+6}$,

$$w(\kappa) = -a_6 - b'\kappa^6 + \mathcal{O}(\kappa^7)\,.$$

To determine the mass M, we return to Eq. (29) and set $p^0 = iM$, getting

$$\begin{aligned} M \;&=\; -\ln\kappa^3 + \ln(1 + c_3(\vec{0})\kappa^3 + w(\kappa)) \\ &=\; -3\ln\kappa + c_3(\vec{0})\kappa^3 - [a_6 + b' + c_3(\vec{0})^2/2]\kappa^6 + \mathcal{O}(\kappa^7)\,, \end{aligned}$$

and the octet-decuplet mass difference is

$$M_d - M_o = (-a_d + a_o)\kappa^6 + \mathcal{O}(\kappa^7) = \frac{3}{4}\kappa^6 + \mathcal{O}(\kappa^7)\,.$$

To end the proof of Theorem 1 , we derive an important analytical property of the octet dispersion curve $w_o(\vec{p})$. Namely, we show the convexity of $w_o(\vec{p})$ at small $|\vec{p}|$. The convexity is easily seen by writing $r_o(\kappa,\vec{p})$ of Eq. (35) as

$$r_o(\kappa,\vec{p}) = r_o(\kappa,\vec{0}) + s_o(\kappa,\vec{p})\,,$$

and noting that the difference $s_o(\kappa,\vec{p}) = r_o(\kappa,\vec{p}) - r_o(\kappa,\vec{0}) = \kappa^6 s(\kappa,\vec{p})$, where s is jointly analytic and bounded by quadratic terms $|p^i|\,|p^j|$, using parity. Thus, the $p_\ell^2\kappa^3/8$ term of $w_o(\vec{p})$ of Eq. (35) dominates and is convex for small $|\vec{p}|$.

In closing the section, we remark that the use of $\mathcal{PC}$T symmetry is enough to show that the two-point function is spin diagonal in simpler models as for free fermions and for the Gross-Neveu model (see Ref. [52]). The determination of dispersion curves for these cases can be done applying the auxiliary function method as for the octet of baryons treated here. Also, we observe that a non-dynamical group theoretical construction of the Grassmann field particle states of Eqs. (27) and (28), based on the flavor group $\mathrm{SU}(3)_f$ can be given as well. To make the text more fluid, we leave this construction for Appendix C.

III. TOTAL HYPERCHARGE, TOTAL ISOSPIN AND SPIN FLIP OPERATORS IN THE PHYSICAL HILBERT SPACE $\mathcal{H}$

In this section, we construct the total hypercharge, total isospin and spin flip operators acting on the physical Hilbert space $\mathcal{H}$. We treated these operators in terms of correlations in Appendix A, where the group generators and notation is given. The spin operator we define is *not* the one from the continuum quantum field theory, but the spin labels we identify with the Grassmann baryon particle basis vectors are the same as those of the continuum case.

We explicitly consider the case of the third component I_3 of isospin. The operators associated with the other $\mathrm{SU}(3)_f$ generators are constructed similarly. The spin flip symmetry and its implementation by an anti-unitary operator is also treated. The spin operators are defined on the field algebra but not in the physical Hilbert space.

For a function F on the field algebra and $U \in \mathrm{SU}(3)$ we define a linear operator $\mathcal{W}(U)$ by (suppressing the gauge variables)

$$\mathcal{W}(U)F = F(\{\psi U^\dagger\},\{U\bar{\psi}\})\,. \tag{36}$$

Using the F-K formula, and for functions F and G of the basic fields, of finite support, we define the Hilbert space operator $\check{\mathcal{W}}(U)$ by the (sesquilinear) form

$$(G, \check{\mathcal{W}}(U)F)_{\mathcal{H}} = \langle [\mathcal{W}(U)F]\Theta G\rangle\,,$$

so that, using Eq. (36), we have

$$
\begin{aligned}
(G, \check{\mathcal{W}}(U)F)_{\mathcal{H}} &= \langle F(\{\psi U^\dagger\},\{U\bar\psi\})\Theta G(\{\psi\},\{\bar\psi\})\rangle \\
&= \langle F(\{\psi\},\{\bar\psi\}\Theta G(\{\psi U\},\{U^\dagger\bar\psi\})\rangle \\
&= \langle F\Theta\mathcal{W}(U^\dagger)G\rangle \\
&= (\check{\mathcal{W}}(U^\dagger)G, F)_{\mathcal{H}}\,,
\end{aligned}
\tag{37}
$$

so that $\check{\mathcal{W}}(U)^\dagger = \check{\mathcal{W}}(U^\dagger)$, and where we have used the $\mathrm{SU}(3)_f$ symmetry on the correlation functions in the rhs.

Furthermore, we have

$$
\begin{aligned}
(\check{\mathcal{W}}(U)G, \check{\mathcal{W}}(U)F)_{\mathcal{H}} &= \langle[\mathcal{W}(U)F]\Theta[\mathcal{W}(U)G]\rangle \\
&= \langle F\Theta G\rangle \\
&= (G,F)_{\mathcal{H}}\,.
\end{aligned}
$$

Hence, $\check{\mathcal{W}}(U)$ is an isometry, i.e. $\check{\mathcal{W}}(U)^\dagger\check{\mathcal{W}}(U)=1$. Interchanging U and $U^\dagger$, we also have $\check{\mathcal{W}}(U)\check{\mathcal{W}}(U)^\dagger = 1$. Then, $\check{\mathcal{W}}(U)^\dagger$ is also isometric which implies that $\check{\mathcal{W}}$ is unitary.

The isometry property of $\check{\mathcal{W}}(U)$ is seen by first considering F and G monomials and then extending to F and G elements of $\mathcal{H}$ by continuity. That $\check{\mathcal{W}}$ in Eq. (37) is well defined follows from the fact that taking $F = G \in \mathcal{N}$ (recall $\mathcal{N}$ denotes the set of nonzero F such that $\langle F\Theta F\rangle = 0$) then if $F\in\mathcal{N}$, $\mathcal{W}(U)F$ is also in $\mathcal{N}$.

We see that $\check{\mathcal{W}}(U)$ commutes with time evolution $\check{T}_0^{x^0}$ by noting that $\mathcal{W}(U)T_0^{x^0}F = T_0^{x^0}\mathcal{W}(U)F$. Thus, the $\mathrm{SU}(3)_f$ generators defined below also commute with $\check{T}_0^{x^0}$.

Taking

$$
U = \exp(i\theta i_3) \in \mathrm{SU}(3)_f\,,
$$

$i_3 \equiv F_3 = \lambda_3/2$, we define $\check{I}_3$ by

$$
(G,\check{I}_3F)_{\mathcal{H}} = \lim_{\theta\to 0}\langle\frac{1}{i\theta}\,\left[F(\{\psi U^\dagger\},\{U\bar\psi\}) - F(\{\psi\},\{\bar\psi\})\right]\,\Theta G(\{\psi\},\{\bar\psi\})\rangle\,. \tag{38}
$$

The other $\mathrm{SU}(3)_f$ generators $F_i = \lambda_i/2$, $i=1,2,4,...,7$ are defined similarly. The total hypercharge Y is defined with $U = e^{iF_8\theta}$, $F_8 = \lambda_8/2$, and a factor of $2/\sqrt{3}$ in front. For $\bar b_{\vec\alpha\vec f}$, $Y = y\times 1\times 1 + 1\times y\times 1 + 1\times 1\times y$, where $y = \mathrm{diag}(1/3, 1/3, -2/3)$.

We consider the special case where we take the Grassmann elements $F = \bar B_{r_2}$ and $G = B_{r_1}$, where $\bar B_{r_1}$ and $\bar B_{r_2}$ are one of the particle basis baryon states of Section II C. Thus, we have

$$
\begin{aligned}
(B_{r_1},\check{I}_3B_{r_2})_{\mathcal{H}} &= \langle I_3\bar B_{r_2}\Theta\bar B_{r_1}\rangle \\
&= (\check{I}_3B_{r_1}, B_{r_2})_{\mathcal{H}}
\end{aligned}
\tag{39}
$$

where $I_3 = i_3\times 1\times 1 + 1\times i_3\times 1 + 1\times 1\times i_3$.

From Eq. (39), we see that $\check{I}_3$ restricted to the one-baryon subspace is self-adjoint.

We now turn to the spin flip symmetry. From a consideration of the composition of the symmetries of time reversal $\mathcal{T}$, charge conjugation $\mathcal{C}$ and time reflection T given in Appendix A, we define the local spin flip operator

$$
\mathcal{F}_s = -i\mathcal{T}\mathcal{C}\mathrm{T}\,, \tag{40}
$$

which acts on single Fermi fields by

$$
\begin{aligned}
\psi_\alpha(x) &\to A_{\alpha\rho}\psi_\rho(x) \\
\bar\psi_\beta(x) &\to \bar\psi_\gamma(x)B_{\gamma\beta}\,,
\end{aligned}
$$

where $A = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2\end{pmatrix}$ is real anti-symmetric and $B = A^{-1} = -A$. For functions of the gauge fields, $f(g_{xy}) \to \bar f(g^*_{xy})$, where * denotes the complex conjugate. More explicitly, $\hat\psi_1\to\hat\psi_2$, $\hat\psi_2\to -\hat\psi_1$, $\hat\psi_3\to\hat\psi_4$ and $\hat\psi_4\to-\hat\psi_3$.

In more detail, if F is a polynomial (not necessarily local) in the gauge and Fermi fields, and suppressing the lattice site arguments,

$$
F = \sum\, a_{\ell mn}\, g^\ell\bar\psi^m\psi^n\,,
$$

then extend $\mathcal{F}_s$ by

$$\mathcal{F}_s F = \sum a^*_{\ell mn}\, g^\ell (\bar\psi B)^m (A\psi)^n\,,$$

and take $\mathcal{F}_s$ to be order preserving.

Note that $\mathcal{F}_s$ is anti-linear but not in the same sense as Θ (no complex conjugation of g for $\mathcal{F}_s$).

With these definitions, the action of Eq. (3) is *termwise invariant* and the symmetry operation is a symmetry of the system satisfying $\langle F\rangle = \langle \mathcal{F}_s F\rangle^*$.

For the 4×4 block of G of the decuplet, applying $\mathcal{F}_s$ gives the structure (with $1,2,3,4$ labelling spin $J_z = 3/2, -3/2, 1/2, -1/2$, respectively)

$$\begin{pmatrix} a & e & c & d \\ -\bar e & \bar a & \bar d & -\bar c \\ f & g & b & h \\ \bar g & -\bar f & -\bar h & \bar b \end{pmatrix}.$$

On the other hand, by $\mathcal{TP}$, the matrix is self-adjoint. Combining $\mathcal{F}_s$ and $\mathcal{TP}$ gives the structure, with $a,b\in\mathbb{R}$,

$$\begin{pmatrix} a & 0 & c & d \\ 0 & a & \bar d & -\bar c \\ \bar c & d & b & 0 \\ \bar d & -c & 0 & b \end{pmatrix}.$$

For the 2×2 block of the octet $\mathcal{F}_s$ gives the structure (with $1,2$ labelling spin $J_z = 1/2, -1/2$, respectively)

$$\begin{pmatrix} q & r \\ -\bar r & q \end{pmatrix}.$$

Again, $\mathcal{TP}$ gives self-adjointness, so that $\mathcal{TP}$ and $\mathcal{F}_s$ yields the structure of the real multiple of the identity.

Alternatively, applying $\mathcal{PC}$T, followed by a translation, gives the above final octet and decuplet matrix structure, directly as in Appendix B.

Now, we implement the spin flip transformation by an anti-unitary operator $\check{\mathcal{F}}_s$ on $\mathcal{H}$. We recall that an anti-unitary transformation $\check{\mathcal{A}}$ on Hilbert space vectors satisfies (see Ref. [54])

$$(\check{\mathcal{A}}G, \check{\mathcal{A}}F)_{\mathcal{H}} = (G,F)^*_{\mathcal{H}}\,.$$

Define $\check{\mathcal{F}}_s$ by

$$(G, \check{\mathcal{F}}_s F)_{\mathcal{H}} = \langle [\mathcal{F}_s F]\Theta G\rangle\,.$$

Then, by direct calculation, using the spin flip symmetry at the level of correlation functions, we have

$$\begin{aligned}(\check{\mathcal{F}}_s G, \check{\mathcal{F}}_s F)^*_{\mathcal{H}} &= \langle [\mathcal{F}_s F]\Theta \mathcal{F}_s G\rangle^* \\ &= \langle F\Theta G\rangle \\ &= (G,F)_{\mathcal{H}}\,.\end{aligned}$$

This shows that $\check{\mathcal{F}}_s$ is well defined, such as, if $F\in\mathcal{N}$ then $\mathcal{F}_s F\in\mathcal{N}$, so that $\check{\mathcal{F}}_s$ can be lifted to $\mathcal{H}$ as an anti-unitary operator.

In general, associated with each of the eight one-parameter subgroups of $\mathrm{SU}(3)_f$ given by $U_j = e^{i\lambda_j\theta/2}$, where $\lambda_{j=1,\ldots,8}$ are the Gell'Mann matrices of Appendix A, is (by Stone' s Theorem) the self-adjoint operator

$$\check{A}_j = \mathrm{s}-\lim_{\theta\to 0} \frac{1}{i\theta}\,[\check{W}(U_j) - 1]\,,$$

acting in $\mathcal{H}$. $\check{I}_{j=1,2,3}$, the j-th component of total isospin is identified with $\check{A}_{j=1,2,3}$. $\check{I}_1$, $\check{I}_2$ and $\check{I}_3$, and $\check{I}^2 = \check{I}_1^2 + \check{I}_2^2 + \check{I}_3^2$, obey the usual angular momentum commutation relations. The total hypercharge operator is $\check{Y} = 2\check{A}_8/\sqrt{3}$ and the quadratic Casimir operator is $\check{C}_2 = \sum_{j=1,\ldots,8} \check{A}_j^2$. $\check{I}_3$, $\check{I}^2$, $\check{Y}$ and $\check{C}_2$ are mutually commuting self-adjoint operators and the eightfold way particle states are conveniently and conventionally labelled by the eigenvalues I_3, I [$I(I+1)$ is the eigenvalue of $\check{I}^2$], Y and C_2, as well as total spin labels J_z and J^2 which we explain next.

We now give the definition of the spin operators on the Grassmann field algebra; **????** they are *not* defined in the physical Hilbert space $\mathcal{H}$. Define the linear operator $Z(U)F$ as in Eq. (36). J_z, the z-component of total spin, is defined by, with $U = U_2 \oplus U_2$, $U_2 = \exp(i\theta\sigma^3/2) \in \mathrm{SU}(2)$,

$$J_z F = \lim_{\theta \to 0} \frac{1}{i\theta}\,[Z(U) - 1]F\,. \tag{41}$$

The other spin components are defined similarly and the total spin square is, of course, $J^2 \equiv \vec{J}^2 = J_x^2 + J_y^2 + J_z^2$. We have not been able to show the well-defined condition for the spin operators so that they are not defined in $\mathcal{H}$.

We recall the definition of the spin operator in the continuum. In the continuum, we identify the components of the total angular momentum with the generators of infinitesimal rotations about coordinate axis. Starting from a state $\Phi(x)$ created by local single or composite field, we consider a zero spatial momentum improper state $\Phi_0 = \int \Phi(x) d\vec{x}$. Φ_0 is expected to have zero spatial angular momentum, i.e. only spin angular momentum, and the rotation operator reduces to a rotation in spin space. The spin operators are defined as the generators of the infinitesimal rotations.

We now turn to definition of the angular momentum operators in our lattice model and make a connection with spin operators defined in Eq. (41). On the lattice, a $\pi/2$ rotation about any one of the coordinate axes x, y, z is a symmetry giving rise to a unitary operator on the Hilbert space $\mathcal{H}$

$$\check{\mathcal{Y}}(U) = \int_{-\pi}^{\pi} e^{i\lambda} dE(\lambda) \tag{42}$$

which is a lift form the linear transformation on functions on the Grassmann algebra such that

$$\mathcal{Y}(U) = F(\{U\psi\}(x_r), \{\bar{U}\psi\}(x_r)) \tag{43}$$

with $x_r = (x^0, \vec{x}_r)$ denoting the coordinates of the rotated point and $U = U_2 \oplus U_2 \equiv e^{i\theta j}$, where $U_2 = e^{i\frac{\theta}{2}\sigma_{x,y,z}}$, $\theta = \pi/2$ and $\sigma_{x,y,z}$ are Pauli matrices. By the spectral theorem, we can define $\check{M}_{x,y,z}$, the components of a lattice total angular momentum, by

$$\check{M} = \frac{2}{i\pi} \ln \check{\mathcal{Y}}(U) = \frac{2}{i\pi} \int_{-\pi}^{\pi} \lambda dE(\lambda)\,. \tag{44}$$

If we consider the zero spatial momentum improper state $\Phi_0 = \sum_{\vec{x}} \Phi(\vec{x})$ (expected to have only spin angular momentum) for the special case of the elementary excitation field of Eq. (1), suppressing indices,

$$\Phi(x) = \epsilon\bar{\psi}\bar{\psi}\bar{\psi}(x)$$

then

$$\mathcal{Y}(U)\Phi_0 = \sum_{\vec{x}} \epsilon\,(U\bar{\psi})(U\bar{\psi})(U\bar{\psi})(x)\,.$$

Intuitively, since we expect that there is no orbital angular momentum, only the internal spin angular momentum survives. We define $J_{x,y,z}$, the components of total spin, acting on Φ_0 by

$$J\Phi_0 = \frac{2}{i\pi} \ln \check{\mathcal{Y}}(U)\Phi_0 = \sum_{\vec{x}} \epsilon\,[(j\bar{\psi})(\bar{\psi})(\bar{\psi}) + (\bar{\psi})(j\bar{\psi})(\bar{\psi}) + (\bar{\psi})(\bar{\psi})(j\bar{\psi})](x)\,, \tag{45}$$

where we have used the spectral theorem with $U = e^{i(\pi/2)j} = \sum_\lambda e^{i(\pi/2)\lambda} P_\lambda$ and, for a function $f(w) = \sum_n a_n w^n$, we have

$$f(\mathcal{Y}(U))\Phi_0 = \sum_{\vec{x}} \epsilon \sum_{\lambda_1,\lambda_2,\lambda_3} f\left(e^{i(\pi/2)(\lambda_1+\lambda_2+\lambda_3)}\right) P_{\lambda_1}\bar{\psi} P_{\lambda_2}\bar{\psi} P_{\lambda_3}\bar{\psi}\,.$$

For f a logarithmic function in this equation, the argument of the logarithm is well-defined for $|\lambda_1 + \lambda_2 + \lambda_3|\pi/2 < \pi$. But this condition is verified since $|\lambda_j| = 1/2$. On the other hand, $J\Phi_0$ in Eq. (45) is precisely what we obtain form the continuous rotation limit

$$\lim_{\theta \searrow 0} \frac{(Z(U) - 1)}{i\theta}\Phi_0\,,$$

where $Z(U)F = F(\{U\bar{\psi}\}, \{\bar{U}\psi\})$, and $Z(U)$ is exactly the transformation appearing in Eq. (41).

The operators $J_{x,y,z}$ obey the usual angular momentum algebra. We suspect that the correlations for different spin states of zero momentum states, within a member of the decuplet, are related by the usual raising and lowering operations. If this is so then the masses of the $J_z = 3/2$ and $J_z = 1/2$ spin states are equal.

IV. UPPER GAP PROPERTY AND UNIQUENESS OF THE BARYON SPECTRUM UP TO THE MESON-BARYON THRESHOLD

Up to now, we have shown that the spectrum generated by the one-baryon and one-antibaryon fields has upper mass gap up to near the meson-baryon threshold of $\simeq -5\ln\kappa$. Here we extend this result from the one-baryon subspace $\mathcal{H}_b$ to the whole odd subspace $\mathcal{H}_o$, adapting the subtraction method of e.g. Refs. [35, 55]. Starting with $L \in \mathcal{H}_o$, from the F-K formula, for $v^0 \neq u^0$, we have

$$\left(L\,,\,(\check{T}^0)^{|v^0-u^0|-1}L\right)_{\mathcal{H}} \;=\; -G(u,v)$$

where, for $M = \Theta L$,

$$G(u,v) = \mathcal{G}_{ML}(u^0,v^0)\chi_{u^0\leq v^0} + \mathcal{G}_{ML}(-u^0,-v^0)\chi_{u^0>v^0}\,,$$

and where we define the general correlation, for all u and v, by

$$\mathcal{G}_{KH}(u,v) = \langle K(u)H(v)\rangle\,. \tag{46}$$

We define the subtracted function (with a summation over $i = 1,2$ understood)

$$F(u,v) = G(u,v) - [R_i\Lambda_i S_i](u,v)\,, \tag{47}$$

with $R_i(u,w) = \mathcal{G}_{M\phi_i}(u^0,w^0)\chi_{u^0\leq v^0} + \mathcal{G}_{M\phi_i}(-u^0,-w^0)\chi_{u^0>v^0}$, $J_i(w,z) = \mathcal{G}_{\rho_i\phi_i}(w^0,z^0)\chi_{w^0\leq z^0} + \mathcal{G}_{\rho_i\phi_i}(-w^0,-z^0)\chi_{w^0>z^0}$, $\Lambda_i = J_i^{-1}$ and $S_i(z,v) = \mathcal{G}_{\rho_i L}(z^0,v^0)\chi_{z^0\leq v^0} + \mathcal{G}_{\rho L}(-z^0,-v^0)\chi_{z^0>v^0}$, where we set $\phi = (\phi_1,\phi_2) = (\bar{b}^\ell, -b^u)$, $\varrho = (\varrho_1,\varrho_2) = (b^\ell, \bar{b}^u)$, with superscripts for the upper and lower spin component written explicitly and where the $\hat{b}$'s are normalized and with the collective indices suppressed.

By time reversal (see Appendix A),

$$\begin{aligned}
\mathcal{G}_{M\phi_i}(-u^0,-w^0) &= -[\mathcal{G}_{L\rho_i}(u^0,w^0)]^*\,,\\
\mathcal{G}_{\rho_i L}(-z^0,-v^0) &= -[\mathcal{G}_{\phi_i M}(z^0,v^0)]^*\,,\\
\mathcal{G}_{\rho_i\phi_i}(-w^0,-z^0) &= -[\mathcal{G}_{\phi_i\rho_i}(w^0,z^0)]^*\,,\\
\mathcal{G}_{ML}(-u^0,-v^0) &= -[\mathcal{G}_{LM}(u^0,w^0)]^*\,.
\end{aligned}$$

We now explain the strategy. The G term in the rhs of Eq. (47) has a spectral representation for unequal times which arises from inserting the spectral representation for $\check{T}^0$ on the lhs of Eq. (46). Its Fourier transform may have singularities in $|\mathcal{I}\mathrm{m}\,p^0| < -(5-\epsilon)\ln\kappa$. The idea is to show that $F^{r=0,1,2,3,4} = 0$ using the hyperplane decoupling method, at least for well separated temporal points, which implies that $\tilde{F}(p)$ has no singularity in $|\mathcal{I}\mathrm{m}p^0| < -(5-\epsilon)\ln\kappa$. Hence, the possible singularities of $\tilde{G}$ are cancelled by those of the Fourier transform of the second term in the rhs of Eq. (47) $\tilde{R}_i\tilde{\Lambda}_i\tilde{S}_i$.

We show that $\tilde{R}_i$ and $\tilde{S}_i$ have spectral representations with the only possible singularities at the spectrum of the eightfold way baryons in $\mathcal{H}_b$. Concerning $\tilde{\Lambda}_i$, noting that J_i is the two-baryon function for $i = 1$, and antibaryon for $i = 2$, the inverse $\tilde{\Lambda}_i$ is analytic in $|\mathcal{I}\mathrm{m}\,p^0| < -(5-\epsilon)\ln\kappa$. Therefore, the singularities of $\tilde{G}$ are precisely those of the eightfold way baryons.

To implement this idea, first we derive a spectral representation for $\mathcal{G}_{M\phi_i}$. For $\mathcal{G}_{\rho_i L}$, a spectral representation is obtained in a similar way. For $i = 1$, as $\phi_1 = \bar{b}^\ell$, using the above definition and suppressing the spatial site indices

$$\mathcal{G}_{M\bar{b}^\ell}(u,v) = \langle T_0^{u^0+1/2}M(-1/2)T_0^{v^0-1/2}\bar{b}^\ell(+1/2)\rangle = -\left(L(+1/2), \check{T}_0^{|v^0-u^0|-1}\bar{b}^\ell(1/2)\right)_{\mathcal{H}}\,,\; u^0 < v^0\,,$$

where we used F-K to write the last equality. The same holds for $u^0 > v^0$. For $i = 2$, $\phi_2 = b^u$, replace $\bar{b}^\ell$ by $-b^u$ on the rhs A spectral representation follows by inserting the spectral representation for $\check{T}_0$.

Similarly, for $i = 1$, $\rho_i = b^\ell$, and

$$\mathcal{G}_{b^\ell L}(u,v) = -\left(\bar{b}^\ell(+1/2), \check{T}_0^{|v^0-u^0|-1}L(+1/2)\right)_{\mathcal{H}}\,,\; u^0 < v^0\,,$$

and the same for $u^0 > v^0$. For $i = 2$, $\rho_2 = \bar{b}^u$, replace $\bar{b}^\ell$ by b^u on the rhs.

Before giving the proof, we give important support properties of the $R_i^{(0)} \equiv R_i^0$, $J_i^{(0)} \equiv J_i^0$, $S_i^{(0)} \equiv S_i^0$ kernels. Here, as before, the upper index ^{n} means the n-th order Taylor series coefficient about $\kappa_p = 0$ of κ_p^n and we suppress the parentheses for simplicity of notation. Furthermore, Λ_i^0 has the same support properties as J_i^0. Namely, by inspection, $J_i^0(w,z)=0$ for $w^0\le p$, $z^0\ge p+1$ and also for $w^0\ge p+1$, $z^0\le p$. Similar properties hold for R_i^0, S_i^0.

We now consider the derivatives at $\kappa_p=0$ of F. By explicit calculation

$$\begin{aligned}F &= [G^0-R^0\Lambda^0S^0]\kappa_p^0+[G^1-R^1\Lambda^0S^0-R^0\Lambda^1S^0-R^0\Lambda^0S^1]\kappa_p\\ &\quad+[G^2-R^2\Lambda^0S^0-R^0\Lambda^2S^0-R^0\Lambda^0S^2-R^1\Lambda^1S^0-R^1\Lambda^0S^1-R^0\Lambda^1S^1]\kappa_p^2\\ &\quad+[G^3-R^3\Lambda^0S^0-R^0\Lambda^3S^0-R^0\Lambda^0S^3-R^2\Lambda^1S^0-R^2\Lambda^0S^1-R^0\Lambda^2S^1\\ &\quad-R^0\Lambda^1S^2-R^1\Lambda^2S^0-R^1\Lambda^0S^2-R^1\Lambda^1S^1]\kappa_p^3+\mathcal{O}(\kappa_p^4)\,.\end{aligned}$$

Here, Λ^r is given by Leibniz formula or by direct calculation from $J\Lambda=1$, and we find

$$\begin{aligned}\Lambda^0 &= (J^0)^{-1}\,,\\ \Lambda^1 &= -\Lambda^0J^1\Lambda^0\,,\\ \Lambda^2 &= \Lambda^0J^1\Lambda^0J^1\Lambda^0-\Lambda^0J^2\Lambda^0\,,\\ \Lambda^3 &= -\Lambda^0J^1\Lambda^0J^1\Lambda^0J^1\Lambda^0+\Lambda^0J^2\Lambda^0J^1\Lambda^0+\Lambda^0J^1\Lambda^0J^2\Lambda^0-\Lambda^0J^3\Lambda^0\,.\end{aligned}$$

For $u^0\le p\le v^0$, by imbalance of fermions and/or by interhyperplane gauge integration, the support properties of J^1 and J^2 are the same as for J^0. The same holds for $G^{1,2}$, $R^{1,2}$, $S^{1,2}$ and $\Lambda^{1,2}$.

Using this, $F^{0,1,2}=0$ and

$$F^3=G^3-R^3\Lambda^0S^0-R^0\Lambda^3S^0-R^0\Lambda^0S^3\equiv A_1+A_2+A_3+A_4\,, \tag{48}$$

for $u^0\le p\le v^0$.

From the general formula $\mathcal{G}^3_{YW}(u,v)=-[\mathcal{G}^0_{Y\phi_i}\circ\mathcal{G}^0_{\rho_iW}](u,v)$, we have

$$\begin{aligned}\mathcal{G}^3_{ML}(u,v) &= -[\mathcal{G}^0_{M\phi_i}\circ\mathcal{G}^0_{\rho_iL}](u,v)\,,\\ \mathcal{G}^3_{M\phi_i}(u,v) &= -[\mathcal{G}^0_{M\phi_j}\circ\mathcal{G}^0_{\rho_j\phi_i}](u,v)\,,\\ \mathcal{G}^3_{\rho_iL}(u,v) &= -[\mathcal{G}^0_{\rho_i\phi_j}\circ\mathcal{G}^0_{\rho_jL}](u,v)\,,\\ \mathcal{G}^3_{\rho_i\phi_i}(u,v) &= -[\mathcal{G}^0_{\rho_i\phi_j}\circ\mathcal{G}^0_{\rho_j\phi_i}](u,v)\,.\end{aligned}$$

We use these relations in Eq. (48). We note that $A_1=\mathcal{G}^3_{ML}=-[\mathcal{G}^0_{M\phi_i}\circ\mathcal{G}^0_{\rho_iL}]$. Using the support properties of R_i^0 and Λ_i^0,

$$\begin{aligned}A_2(u,v) &= -\textstyle\sum_{z^0,w^0\le p}R_i^0(u,w)\Lambda_i^0(w,z)[-\mathcal{G}^0_{\rho_i\phi_i}\circ\mathcal{G}^0_{\rho_iL}](z,v)\\ &= R_i^0(u,p)\mathcal{G}^0_{\rho_iL}(p+1,v)\\ &= -[\mathcal{G}^0_{M\phi_i}\circ\mathcal{G}^0_{\rho_iL}](u,v)\\ &= -A_1(u,v)\,.\end{aligned}$$

We have extended the z sum in the first equality and used $\Lambda_i^0J_i^0=1$. A_4 is treated similarly and gives $A_4=A_2=-A_1$.

Next, by the support properties of R_i^0 and S_i^0, and using $\Lambda_i^3=-\Lambda_i^0J_i^3\Lambda_i^0$, $w^0\le p$, $z^0\ge p+1$, and noticing that $J_i^3(u,v)=-[\mathcal{G}^0_{\rho_i\phi_i}\circ\mathcal{G}^0_{\rho_i\phi_i}](u,v)$, for $u^0\le p$, $v^0\ge p+1$,

$$A_3(u,v)=-\sum_{w^0,x^0\le p,z^0,y^0\ge p+1}R_i^0(u,w)\Lambda_i^0(w,x)[\mathcal{G}^0_{\rho_i\phi_i}\circ\mathcal{G}^0_{\rho_i\phi_i}](x,y)\Lambda_i^0(y,z)S_i^0(z,v)\,.$$

Then, extending the sum on x^0 and y^0, we obtain

$$A_3(u,v)=-R_i^0(u,p)S_i^0(p+1,v)=-[\mathcal{G}^0_{M\phi_i}\circ\mathcal{G}^0_{\rho_iL}](u,v)=A_1\,.$$

Thus, we have $F^3(u,v)=0$, for $u^0<v^0$

For the other temporal order $u^0>p\ge v^0$, the treatment is similar but more delicate.

Here we use $\mathcal{G}^3_{YW}(u,v)=[\mathcal{G}^0_{Y\phi_i'}\circ\mathcal{G}^0_{\rho_i'W}](u,v)$, where $\phi'=(b^\ell,-b^u)$, $\rho'=(\bar{b}^\ell,b^u)$. If we set, as before, $F^3=\sum_{k=1,2,3,4}A_k'$, we note that

$$A_1'=-[\mathcal{G}^0_{L\rho_i}]^*\circ[\mathcal{G}^0_{\phi_iM}]^*\,.$$

For A_2', we find

$$A_2' = \sum_{w^0, z^0 > p} R_i^0(u,w)\Lambda_i^0(w,z)[\mathcal{G}^0_{\phi_i\rho_i}(z,p+1)]^*[\mathcal{G}^0_{\phi_i M}(p,v^0)]^* \,.$$

Roughly speaking, we want to replace $[\mathcal{G}^0_{\phi_i\rho_i}(z,p+1)]^*$ by $-J_i^0(z,p+1)$ and use $\Lambda_i^0 J_i^0 = 1$. We write

$$\begin{aligned}[\mathcal{G}^0_{\phi_i\rho_i}(z,p+1)]^*\chi_{z^0>p} &= [\mathcal{G}^0_{\phi_i\rho_i}(z,p+1)]^*\chi_{z^0>p+1} - \mathcal{G}^0_{\rho_i\phi_i}(z,p+1)\chi_{z^0\le p+1} \\ &\quad + [\mathcal{G}^0_{\phi_i\rho_i}(z,p+1)]^*\chi_{z^0=p+1} + [\mathcal{G}^0_{\rho_i\phi_i}(z,p+1)]^*\chi_{z^0=p+1}\end{aligned} \tag{49}$$

under the w^0, z^0 summation.

Using Eq. (49), the first two terms of A_2' are

$$-R_i^0(u,p+1)[\mathcal{G}_{\phi_i M}(p,v)]^* = \{[\mathcal{G}^0_{L\rho_i}]^* \circ [\mathcal{G}^0_{\phi_i M}]^*\}(u,v) \,,$$

for $u^0 > p+1$. Hence $A_2' = -A_1'$, for $u^0 > p+1$.

For the last two terms in Eq. (49), we get zero by time reversal and anti-commutation.

The terms A_3' and A_4' are treated similarly to give $A_4' = A_2' = -A_1'$ and, for $u^0 > p+1$ and $v^0 < p$, $A_3' = A_1'$. Finally $F^3(u,v) = 0$, for $u^0 > p+1$, $v^0 < p$.

Summarizing, $F^3(u,v) = 0$, if $|u-v| > 2$.

With this, the upper gap property of Theorem 1 is guaranteed.

V. FINAL REMARKS

We have considered the subspace of the physical Hilbert space with an odd number of fermions with the same bare mass and flavor $\mathrm{SU}(3)_f$ symmetry, which has baryons and antibaryons. We find that all the masses are equal for the octet, for all $\kappa > 0$, and for the decuplet up to and including order κ^6.

Our method extends directly to treat rigorously the case where the quark masses are different and there is no flavor symmetry. For example, we can break the octet and the decuplet hypercharge mass degeneracy by making the s quark mass different from the u and d masses. The product structure still holds and the elementary excitations are revealed and are the same $\bar{b}_{\vec{\alpha}\vec{f}}$ of Section II B. We can also analyze the even subspace $\mathcal{H}_e$ where the mesons lie. The analysis of the mesonic sector showing the existence of the eightfold way particles as the only spectrum in $\mathcal{H}_e$ up to near the two-meson threshold of $\simeq -4\ln\kappa$ will appear soon in Ref. [45].

We have used the *new* symmetry of $\mathcal{PCT}$ or, alternatively, a local spin flip symmetry $\mathcal{F}_s$ in our lattice model. We emphasize that, formally, it extends to relativistic field theories as the free fermion and the Gross-Neveu model. It would be interesting to explore this symmetry (and possible generalizations) and the relations for more general correlation functions and its consequence e.g. to consider general higher spin relativistic quantum fields.

It would be interesting to provide a mathematically rigorous construction of dispersion curves for the total spin $3/2$ decuplet baryon states, as we did for the total spin $1/2$ octet baryons.

Whether or not there is mass splitting within the decuplet is a question to be answered. If there is no mass splitting, is it a consequence of some deeper property such as an additional symmetry? Note that, in the continuum limit, if the local baryon field transforms according to the spin $3/2$ representation of the Poincaré group, then by a Lorentz boost we can go to the rest frame, and rotational invariance ensures that the masses are the same for $J_z = 3/2,\, 1/2,\, -1/2,\, -3/2$, by spin lowering. Besides, there is only one dispersion curve, the relativistic one.

Also, from a group theoretical viewpoint, it would also be interesting to determine what representations in the individual basis for $\mathrm{SU}(N_f)_f$ occur. Precisely, considering the lowest dimensional representations $\mathbf{2}$ of the lower spin group $\mathrm{SU}_\ell(2)$ and $\mathbf{N_f}$ of the flavor group $\mathrm{SU}(N_f)_f$, the baryons are given by the tensor product of representations

$$[\mathbf{2}\times\mathbf{N_f}]\times[\mathbf{2}\times\mathbf{N_f}]\times[\mathbf{2}\times\mathbf{N_f}]\,,$$

and we take the symmetric part of it due to the totally symmetric property. We would like to know which representations of $\mathrm{SU}_\ell(2)\times\mathrm{SU}(N_f)_f$ survive.

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APPENDIX A: Symmetry, Correlation Function Orthogonality Relations and Identities

Here we obtain two-baryon correlation function orthogonality relations and identities using $\mathrm{SU}(3)_f$ and other symmetry groups. These identities and relations are extensively used to reduce the two-point function matrix in the particle basis to a block diagonal form with the smallest blocks possible.

For $U \in \mathrm{SU}(3)_f$, we write

$$U = \exp[i\vec{\lambda}.\vec{\theta}/2]\,,$$

where $\vec{\theta} = (\theta_1, ..., \theta_8)$ and $\theta_j \in \mathbb{R}$, and λ_j, $j = 1, ..., 8$ are the traceless self-adjoint Gell'Mann matrices given by

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

There is a distinguished subgroup, isomorphic to SU(2), with generators λ_1, λ_2 and λ_3, which is identified with isospin. By convention, we set $F_i = \lambda_i/2$. We also set $\mathrm{i}_{j=1,2,3} = F_{j=1,2,3}$. The i_j are the isospin components which obey the usual spin 1/2 angular momentum commutation relations $[\mathrm{i}_j, \mathrm{i}_k] = i\epsilon_{jkl}\mathrm{i}_l$; and similar for the isospin raising and lowering operators $\mathrm{i}_\pm = \mathrm{i}_1 \pm i\, \mathrm{i}_2$. Next, we set $y = 2F_8/\sqrt{3}$ which generates a one-parameter subgroup associated with hypercharge; the diagonal elements are associated with the hypercharge of individual quarks. There are two Casimir operators C_2 and C_3, quadratic and trilinear, respectively, in the F_j's, which commute with all the generators F_j. We only use the quadratic Casimir C_2 given by $\sum_j F_j^2$.

Two-baryon correlations are linear combination of the basic correlation

$$\langle \psi_{f_1}\psi_{f_2}\psi_{f_3}\bar{\psi}_{f_4}\bar{\psi}_{f_5}\bar{\psi}_{f_6}\rangle \equiv T_{123;456}\,,$$

where we suppress all but the flavor indices.

We will consider quantities of the type

$$(A\bar{W})_{f_1f_2f_3}T_{f_1f_2f_3f_4f_5f_6}(B\bar{V})_{f_4f_5f_6}\,,$$

where the $v_{f_1f_2f_3}$ and $w_{f_1f_2f_3}$ are components of a vector in the three-fold product space $\mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3$, and A and B are linear operators on this space. We want to see what relations the product space operators inherit from relations obeyed by the operators acting in the $\mathbb{C}^3$ factor space, e.g. commutation relations.

In particular, if A has the form

$$A = (a \times 1 \times 1) + (1 \times a \times 1) + (1 \times 1 \times a)\,,$$

and similarly for B and C, then

$$AB = (ab\times 1\times 1) + (1\times ab\times 1) + (1\times 1\times ab) + (a\times b\times 1) + (a\times 1\times b) + (1\times a\times b) + (b\times a\times 1) + (b\times 1\times a) + (1\times b\times a)\,,$$

where we note the symmetry in a and b in the last six terms. For the commutator in the product space, we have $[A, B]_3 = C$ if $[a, b]_1 = c$ in the factor space. From now on, we drop the subscript on the commutators whenever there is no confusion.

We start from the identity, with arbitrary $U \in \mathrm{U}(3)$,

$$\langle \psi_{f_1}\psi_{f_2}\psi_{f_3}(\bar{\psi}\,U)_{f_4}(\bar{\psi}\,U)_{f_5}(\bar{\psi}\,U)_{f_6}\rangle = \langle (U\,\psi)_{f_1}(U\,\psi)_{f_2}(U\,\psi)_{f_3}\bar{\psi}_{f_4}\bar{\psi}_{f_5}\bar{\psi}_{f_6}\rangle\,,$$

where the rhs is obtained from the lhs using the three-flavor isospin symmetry. We schematically rewrite this relation as

$$(TU_3)_{123;456} = (U_3T)_{123;456}\,, \tag{A1}$$

where we set $U_3 \equiv U \otimes U \otimes U$.

We obtain orthogonality relations for the two-baryon correlation by relating this identity to the usual quantum mechanical sum of angular momentum relations. We write $U = e^{i\theta\lambda/2}$, $\mathrm{i} = \lambda/2$, where $\theta\lambda = \theta_1\lambda_1,\, \theta_2\lambda_2,\, \theta_3\lambda_3$, with $\theta_{1,2,3} \in \mathbb{R}$. Expanding in θ, we have $U = 1 + i\theta\mathrm{i} - \frac{\theta^2}{2}\mathrm{i}^2 + \ldots$ such that

$$\begin{aligned} U_3 = &\ 1 \otimes 1 \otimes 1 + i\theta(\mathrm{i} \otimes 1 \otimes 1 + 1 \otimes \mathrm{i} \otimes 1 + 1 \otimes 1 \otimes \mathrm{i}) \\ &-\frac{\theta^2}{2}\,(\mathrm{i}^2 \otimes 1 \otimes 1 + 1 \otimes \mathrm{i}^2 \otimes 1 + 1 \otimes 1 \otimes \mathrm{i}^2 \\ &+ 2\,\mathrm{i} \otimes \mathrm{i} \otimes 1 + 2\,\mathrm{i} \otimes 1 \otimes \mathrm{i} + 2\,1 \otimes \mathrm{i} \otimes \mathrm{i}) \ldots \end{aligned} \tag{A2}$$

Letting w_{123}, v_{123} be components of a vector in the product space $\mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3$, multiplying Eq. (A1) by $\bar{w}_{123}$ on the left and v_{123} on the right, and equating the coefficients of $i\theta_3$, we obtain the identity

$$\bar{w}_{123}(TI_3)_{123;456}v_{456} = \bar{w}_{123}(I_3T)_{123;456}v_{456}\,, \tag{A3}$$

where $I_j = \mathrm{i}_j \otimes 1 \otimes 1 + 1 \otimes \mathrm{i}_j \otimes 1 + 1 \otimes 1 \otimes \mathrm{i}_j$. $\mathrm{i}_j = \lambda_j/2$ is the jth component of individual isospin and we have

$$[\mathrm{i}_+, \mathrm{i}_-] = 2\mathrm{i}_3 \quad , \quad [\mathrm{i}_3, \mathrm{i}_\pm] = \pm\mathrm{i}_\pm\,, \tag{A4}$$

for

$$\mathrm{i}_\pm = \mathrm{i}_1 \pm i\mathrm{i}_2\,. \tag{A5}$$

In a similar way, equating the coefficients of $-(\theta_1^2 + \theta_2^2 + \theta_3^2)/2$ we have the identity

$$\bar{w}_{123}(TI^2)_{123;456}v_{456} = \bar{w}_{123}(I^2T)_{123;456}v_{456}\,,$$

where

$$\vec{I}^2 = I_1^2 + I_2^2 + I_3^2 = (I_+I_- + I_-I_+)/2 + I_3^2\,,$$

and $I_\pm = I_1 \pm iI_2$. I_3 ($\vec{I}^2$) has the interpretation of the third-component (square) of total baryon isospin and we also have

$$[I_+, I_-] = 2I_3 \quad , \quad [I_3, I_\pm] = \pm I_\pm\,. \tag{A6}$$

As $\lambda_{1,2,3}$ are self-adjoint, we can use the common complex Hilbert space notation (,), with the complex conjugate in the first factor, and write the above as

$$(w, TI_3v) = (I_3w, Tv)\ ;\ (w, TI^2v) = (I^2w, Tv)\,. \tag{A7}$$

Thus, if v and w correspond to eigenfunctions with distinct eigenvalues we have orthogonality. To be more explicit, using the usual quantum mechanics notation for α (β) for spin up (down) state, and suppressing the tensor product notation, an isospin state of total isospin $\frac{3}{2}$ and $z-$ component $\frac{3}{2}$ ($\frac{1}{2}$) are represented as the Clebsch-Gordan (C-G) linear combination $v_{\frac{3}{2}\frac{3}{2}} = \alpha\alpha\alpha$ and $v_{\frac{3}{2}\frac{1}{2}} = \frac{1}{\sqrt{3}}(\alpha\alpha\beta + \alpha\beta\alpha + \beta\alpha\alpha)$, respectively.

Taking these states into Eqs. (A7), and using the usual quantum mechanical relations gives the orthogonality relations. Furthermore, taking the appropriate gauge and spin linear combinations gives the orthogonality relations for the two-baryon function.

Similar to the I_3 identity of Eq. (A3), we also have

$$(w, TI_\pm v) = (I_\mp w, Tv)\,. \tag{A8}$$

Next, we take $v = \chi_{Im}$ and $w = \chi_{I(m-1)}$, where $\chi_{I\ell}$ is the usual normalized eigenfunction of total isospin I and $z-$ component ℓ. Hence, $\chi_{I\ell}$ satisfies $I_\pm\chi_{I\ell} = c_\ell^\pm\,\chi_{I(\ell\pm1)}$, where

$$c_\ell^\pm = [I(I+1) - \ell(\ell \pm 1)]^{1/2}\,,$$

and we have used $[I_+, I_-] = 2I_z$ and $I^2 \equiv \vec{I}^2 = I_z^2 + (I_+I_- + I_-I_+)/2$. Substituting in Eq. (A8), with the lower sign on the lhs, and noting that $c_\ell^- = c_{\ell-1}^+$, gives

$$(\chi_{I(\ell-1)}, T\chi_{I(\ell-1)}) = (\chi_{I\ell}, T\chi_{I\ell})\,,$$

which implies the identity

$$\langle \phi_{I(\ell-1)}(x)\bar{\phi}_{I(\ell-1)}(y)\rangle = \langle \phi_{I\ell}(x)\bar{\phi}_{I\ell}(y)\rangle \,. \tag{A9}$$

Of course, Eq. (A9) implies that all associated one-baryon spectral properties are the same, for I fixed, and all I_3.

For the one-parameter hypercharge subgroup associated with F_8, we have the identity

$$(w, TYv) = (Yw, Tv)\,, \tag{A10}$$

where $Y = y \times 1 \times 1 \,+\, 1 \times y \times 1 \,+\, 1 \times 1 \times y$ is the baryon total hypercharge.

Similar to isospin, and following the conventions of Ref. [3], we introduce $u_\pm = F_6 \pm iF_7$, $v_\pm = F_4 \pm iF_5$, $2u_3 = \frac{3}{2}y - \mathrm{i}_3$, $2v_3 = \frac{3}{2}y + \mathrm{i}_3$ and define the associated product space operators $U_\pm$, U_3, $V_\pm$, V_3, where $2U_3 = \frac{3}{2}Y - I_3$ and $2V_3 = \frac{3}{2}Y + I_3$.

The u's and v's obey the same relations as the i's given in Eq. (A4); and the U's and V's obey the same relations as the I's [see Eq. (A6)]. We see that $I_\pm$ changes I_3 by ± 1. In addition, we have, and inherited from the factor relations,

$$[Y, I_\pm] = 0 \;\; ; \;\; [Y, I_3] = 0 \;\; ; \;\; [I_3, U_\pm] = \mp U_\pm/2 \;\; ; \;\; [Y, U_\pm] = \pm U_\pm \;\; ; \;\; [I_3, V_\pm] = \pm V_\pm/2 \;\; ; \;\; [Y, V_\pm] = \pm V_\pm \,. \tag{A11}$$

So, $I_\pm$ changes I_3 by ± 1, but does not change Y; $U_\pm$ changes I_3 by $\mp\frac{1}{2}$ and changes Y by ± 1; $V_\pm$ changes I_3 by $\pm\frac{1}{2}$ and changes Y by ± 1.

We use the operators $I_\pm$ to move within a given isospin multiplet, with the same hypercharge Y and the operators $U_\pm$ and $V_\pm$ to move to different isospin multiplets. Similar correlation relations and identities hold for the U's and V's.

Let us consider a convenient form for the quadratic $\mathrm{SU}(3)_f$ operator C_2. As $F_4^2 + F_5^2 = \frac{1}{2}\left(V_+V_- + V_-V_+\right)$, $F_6^2 + F_7^2 = \frac{1}{2}\left(U_+U_- + U_-U_+\right)$ and, by the commutation relations, $[V_+, V_-] = 2V_3$ and $[U_+, U_-] = 2U_3$, C_2 can be written as

$$C_2 = \vec{I}^{\,2} + V_-V_+ + V_3 + U_-U_+ + U_3 + \frac{3}{4}Y^2 \,.$$

To calculate C_2 acting on octet and decuplet vectors, we note that V_+ and U_+ acting on p of the octet and Δ^{++} of the decuplet vanish. Thus, for these vectors C_2 reduces to

$$C_2 = \vec{I}^{\,2} + U_3 + V_3 + \frac{3}{4}Y^2 \,,$$

where we have used $V_3 = 3Y/4 + I_3/2$ and $U_3 = 3Y/4 - I_3/2$. If we start from a state with a definite value for the product space quadratic Casimir $C_2 = \vec{I}^{\,2} + U_3 + V_3 + \frac{3}{4}Y^2$, for the state with maximum U_3 and V_3 eigenvalues, then all the states generated by applications of $I_\pm$, $U_\pm$, $V_\pm$ have the same value of C_2. For the octet (decuplet)C_2 takes the value 3 (6). Thus, for fixed value of J_z, we can use the eigenvalues of the mutually commuting operators I_3, $\vec{I}^{\,2}$, Y and C_2 to label members of the octet and the decuplet, and they are distinguished by this labelling.

Using the relations obtained up to now reduces the baryon-baryon correlation matrix $G_{r_1 r_2}$ to a block diagonal form with eight identical 2×2 blocks, associated with total spin 1/2 octet baryons, and ten identical 4×4 blocks associated with total spin 3/2 decuplet baryons. To obtain a more diagonal form, we now use the $\mathcal{PC}$T symmetry following Appendix B of Ref. [42]. These symmetries are summarized as follows:

- Time Reflection T: The $\hat{\psi}$ fields transform as $\psi_\alpha(x) \to A_{\alpha\beta}\psi_\beta(-x^0, \vec{x})$, $\bar{\psi}_\alpha(x) \to \bar{\psi}_\beta(-x^0, \vec{x})B_{\beta\alpha}$ where $A = B = A^{-1} = \begin{pmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{pmatrix}$, $f(g_{xy}) \to f(g_{\bar{x}\bar{y}})$, with $\bar{z} = (-z^0, \vec{z})$;
- Charge Conjugation $\mathcal{C}$: $\psi_\alpha(x) \to \bar{\psi}_\beta(x)A_{\beta\alpha}$, $\bar{\psi}_\alpha(x) \to B_{\alpha\beta}\psi_\beta(x)$, $A = -B = B^{-1} = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}$, $f(g_{xy}) \to f(g^*_{xy})$;
- Parity $\mathcal{P}$: $\psi_\alpha(x) \to A_{\alpha\beta}\psi_\beta(x^0, -\vec{x})$, $\bar{\psi}_\alpha(x) \to \bar{\psi}_\beta(x^0, -\vec{x})B_{\beta\alpha}$ where $A = B = A^{-1} = \gamma^0$, $f(g_{xy}) \to f(g_{\bar{x}\bar{y}})$, with $\bar{z} = (z^0, -\vec{z})$.

We remark that the time reflection T symmetry is a new symmetry not to be confused with the usual time reversal symmetry given by

- Time Reversal $\mathcal{T}$: $\psi_\alpha(x) \to \bar\psi_\beta(x^t)A_{\beta\alpha}$, $x^t \equiv (-x^0,\vec{x})$, $\bar\psi_\alpha(x) \to B_{\alpha\beta}\psi_\beta(x^t)$, $A = B = B^{-1} = \gamma^0$, $f(g_{xy}) \to [f(g_{x^ty^t})]^*$.

The above single field symmetry operations extend to monomials and are taken to be order preserving, except for $\mathcal{C}$ and $\mathcal{T}$ which are order reversing. They extend linearly to polynomials and their limits, except for time reversal which is anti-linear. For all of them, except time reversal, the field average equals the transformed field average; for time reversal the transformed field average is the complex conjugate of the field average.

Remark A1 *An important result is that the composed operation $\mathcal{F}_s \equiv -i\mathcal{T}\mathcal{C}\mathrm{T}$ of Eq. (40) gives the spin flip transformation. It is local and is also a symmetry of the system. In contrast to the above transformations, it surprisingly leaves invariant each individual term in the action of Eq. (3).*

Applying the $\mathcal{PC}$ T symmetry, followed by translation, we find the following relations, after making a half-integer shift in the temporal coordinate of the lattice,

$$\langle p_\pm(u)\bar p_\mp(v)\rangle = 0 \quad , \quad \langle p_+(u)\bar p_+(v)\rangle = \langle p_-(u)\bar p_-(v)\rangle \, , \tag{A12}$$

and the same for the n's. We also have

$$\langle p_\pm(u)\bar n_\mp(v)\rangle = 0 = \langle n_\pm(u)\bar p_\mp(v)\rangle \ , \ \text{etc}\, .$$

Setting $\hat\phi_s(u) \equiv \hat B_{\frac32\frac32 s}(u)$, we also have

$$\begin{aligned}
&\langle \phi_{\frac32}(u)\bar\phi_{-\frac32}(v)\rangle = \langle \phi_{\frac12}(u)\bar\phi_{-\frac12}(v)\rangle = 0\,,\\
&\langle \phi_{\mp\frac32}(u)\bar\phi_{\mp\frac12}(v)\rangle = -\langle \phi_{\pm\frac12}(u)\bar\phi_{\pm\frac32}(v)\rangle\,,\\
&\langle \phi_{-\frac32}(u)\bar\phi_{\frac12}(v)\rangle = \langle \phi_{-\frac12}(u)\bar\phi_{\frac32}(v)\rangle\,,\\
&\langle \phi_{\frac32}(u)\bar\phi_{\frac32}(v)\rangle = \langle \phi_{-\frac32}(u)\bar\phi_{-\frac32}(v)\rangle\,,\\
&\langle \phi_{\frac12}(u)\bar\phi_{\frac12}(v)\rangle = \langle \phi_{-\frac12}(u)\bar\phi_{-\frac12}(v)\rangle\,,
\end{aligned}$$

The relations after Eq. (A12) carry over to the two-baryon correlation G.

Using Eq. (A12), the proton and neutron two-point functions are diagonal in spin to all orders in κ. For spin 3/2, label the basis by 3/2, $-3/2$, 1/2, and $-1/2$, or, simply, 1, 2, 3, and 4, respectively. With this labelling, remarking that time reversal $\mathcal{T}$ parity $\mathcal{P}$ and the spectral representation ensures self-adjointness and using the relations after Eq. (A12), the two-point function matrix has the structure

$$\begin{pmatrix} a & 0 & c & d \\ 0 & a & \bar d & -\bar c \\ \bar c & d & b & 0 \\ \bar d & -c & 0 & b \end{pmatrix},$$

$a, b \in \mathbb{R}$, with multiplicity two eigenvalues

$$\mu_\pm = \{(b+a) \pm [(b-a)^2 + 4(|c|^2+|d|^2)]^{1/2}\}/2\,.$$

The corresponding eigenvectors are $(c, \bar d, \mu_\pm - a, 0)$ and $(\mu_\pm - b, 0, \bar c, \bar d)$. Note that these two linearly independent vectors have a singular limit when $c, d \to 0$, where they become identical. [This limit corresponds to taking $\tilde\Gamma_{13}\,, \tilde\Gamma_{14} \to 0$ in Eq. (A14) below.]

The same matrix structure is satisfied by its inverse matrix, as follows from the formula for the inverse matrix. Consequently, it also holds for the Fourier transforms $\tilde G_{ss'}(p^0 = i\chi, \vec p)$ and $\tilde\Gamma_{ss'}(p^0 = i\chi, \vec p)$, $\chi \in \mathbb{R}$. Using the r_3 symmetry given below in Eq. (A15), shows that these matrices are diagonal for $\vec p = \vec 0$ as well as for $\vec p = (p^1 = 0, p^2 = 0, p^3)$, which is used in Ref. [35] to determine the one-baryon mass spectrum and to show that no mass splitting occurs up to and including order κ^6. For $\vec p \neq \vec 0$, recall that the one-baryon dispersion curves $w(\vec p)$ are the solutions of $\det \tilde\Gamma(p^0 = iw(\vec p), \vec p) = 0$. From the above structure, omitting the $\vec p$ dependence, the determinant factorizes as

$$\det\tilde\Gamma = [\lambda_+\lambda_-]^2 \ , \tag{A13}$$

which gives two by two identical dispersion curves associated to the zeroes of

$$\lambda_\pm \equiv \frac12\left[\tilde\Gamma_{11} + \tilde\Gamma_{33}\right] \pm \sqrt{\frac14\left[\tilde\Gamma_{11} - \tilde\Gamma_{33}\right]^2 + \left|\tilde\Gamma_{13}\right|^2 + \left|\tilde\Gamma_{14}\right|^2}\,. \tag{A14}$$

For $\vec{p}=\vec{0}$, $\lambda_+ = \tilde{\Gamma}_{11}$ and $\lambda_- = \tilde{\Gamma}_{33}$, and if they are not equal, mass splitting occurs.

For $\vec{p}\neq\vec{0}$, this is the most diagonal form we have been able to obtain for $\tilde{G}(p^0=i\chi,\vec{p})$ and $\tilde{\Gamma}(p^0=i\chi,\vec{p})$. For $\vec{p}=\vec{0}$, we can reduce these matrices to a completely diagonal form. To do this, use the $\pi/2$ rotation about e^3, the r_3 symmetry, given by

- Rotation r_3 of $\pi/2$ about e^3: $\psi_\alpha(x)\to A_{\alpha\beta}\psi_\beta(x^0,x^2,-x^1,x^3)$, $\bar\psi_\alpha(x)\to\bar\psi_\beta(x^0,x^2,-x^1,x^3)B_{\beta\alpha}$ where $A=B^{-1}=\mathrm{diag}(e^{-i\pi/4},e^{i\pi/4},e^{-i\pi/4},e^{i\pi/4})$, $f(g_{xy})\to f(g_{\bar{x}\bar{y}})$, with $\bar{z}=(z^0,z^2,-z^1,z^3)$.

Using this symmetry, with $x_r\equiv(x^0,x^2,-x^1,x^3)$, we obtain the relations

$$\begin{aligned}
&\langle\phi_{\frac32}(u)\bar\phi_{\frac12}(v)\rangle = -i\langle\phi_{\frac32}(u_r)\bar\phi_{\frac12}(v)\rangle\,,\\
&\langle\phi_{\frac32}(u)\bar\phi_{-\frac12}(v)\rangle = -\langle\phi_{\frac32}(u_r)\bar\phi_{-\frac12}(v)\rangle\,,\\
&\langle\phi_{\frac32}(u)\bar\phi_{-\frac32}(v)\rangle = i\langle\phi_{\frac32}(u_r)\bar\phi_{-\frac32}(v)\rangle\,,\\
&\langle\phi_{\frac12}(u)\bar\phi_{-\frac12}(v)\rangle = -i\langle\phi_{\frac12}(u_r)\bar\phi_{-\frac12}(v)\rangle\,,\\
&\langle\phi_{\frac12}(u)\bar\phi_{-\frac32}(v)\rangle = -\langle\phi_{\frac12}(u_r)\bar\phi_{-\frac32}(v)\rangle\,,\\
&\langle\phi_{-\frac12}(u)\bar\phi_{-\frac32}(v)\rangle = -i\langle\phi_{-\frac12}(u_r)\bar\phi_{-\frac32}(v)\rangle\,,\\
&\langle\phi_s(u)\bar\phi_s(v)\rangle = \langle\phi_s(u_r)\bar\phi_s(v)\rangle\,.
\end{aligned}\tag{A15}$$

These relations carry over to the baryon-baryon correlation function. Taking the Fourier transform and setting $\vec{p}=\vec{0}$, shows that the off-diagonal elements vanish. To relate diagonal elements, we use the e^1 reflection symmetry given by

- Reflection in e^1: $\mathcal{R}_1(e^1,e^2,e^3)=(-e^1,e^2,e^3)$, with $A=i\gamma^0\gamma^2\gamma^3=A^{-1}=B$, $A=\begin{pmatrix}-\sigma^1 & 0\\ 0 & \sigma^1\end{pmatrix}$, and such that $f(g_{xy})\mapsto f(g_{\mathcal{R}_1x\,\mathcal{R}_1y})$.

Reflection in e^1 shows that $\Gamma_{s_1s_2}(x)=\Gamma_{-s_1\,-s_2}(x^0,-x^1,x^2,x^3)$. Thus, $\tilde\Gamma_{s_1s_2}(p^0,\vec{p}=\vec{0})=\tilde\Gamma_{-s_1\,-s_2}(p^0,\vec{p}=\vec{0})$, which gives the structure (a,a,b,b) on the diagonal. This was obtainined above using spin flip symmetry $\mathcal{F}_s$.

In this case, we can apply the auxiliary function method to obtain convergent expansions for the masses, and both masses are equal up to and including $\mathcal{O}(\kappa^6)$.

Apparently, nothing forbids diagonal elements to be nonequal. In Ref. [35], for the one-flavor model in $2+1$ dimensions and 4×4 Dirac matrices, we show that there is mass splitting at order κ^6.

APPENDIX B: Small Distance Behavior of G and Γ

In this Appendix, we prove the short distance behavior for G and Γ. As given before, $\Gamma=G^{-1}$ is obtained from a Neumann expansion using the decomposition of $G=G_d+G_n$ [see Eq. (21)]. Noting that the diagonal part G_d of G satisfies $G_d=-1+\mathcal{O}(\kappa^8)$, to get control of Γ for the points and orders in κ we consider here we can take $G_d=-1$ and then, for calculating to the orders of κ that we have made explicit, Γ can be taken as

$$\Gamma \;=\; -\sum_{r=1}^{6} G_n^r\,.$$

So, we only need to determine G_n. The contributions for $G(x)\equiv G(0,x)$ can be represented by graphs consisting of ordered oriented bonds of unit length. The arrows emerge from the $\bar\psi$'s in the action and associate a $-\kappa/2$ factor each. The arrows have a direction, an isospin index attached to its beginning and another to its end, a spin index attached to its beginning and another to its end, and also associate a $\mathrm{SU}(3)_c$ gauge matrix element. There are sums over the intermediate color, spin and isospin indices using the free fermion propagators. According to the cofactor method (see Ref. [22] and references therein) the only gauge integrals that survive are the ones with a multiple of 3 superposed bonds, each g^{-1} (opposite orientation) counted as two g's. This restriction carries over to the graphs. Each graph contributing to $G(x)\equiv G(0,x)$ has a path of bonds connecting 0 to x. By inspection, for the lowest orders in κ, the only paths that we need can only be built with two superimposed bonds of opposite orientation or three coinciding bonds (same orientation), which give rise to the gauge integrals $\mathcal{I}_2$ and $\mathcal{I}_3$ (see Refs. [22, 35, 36, 42].

Other useful gauge integrals are $\mathcal{I}_4$ and $\mathcal{I}_6$. These gauge integrals are:

$$
\begin{aligned}
\mathcal{I}_2 &\equiv \int U(g)_{a_1b_1}U(g)^{-1}_{a_2b_2}\,d\mu(g) \equiv \int g_{a_1b_1}g^{-1}_{a_2b_2}\,d\mu(g) = \frac{1}{3}\,\delta_{a_1b_2}\,\delta_{a_2b_1}\,, &\text{(B1)}\\
\mathcal{I}_3 &\equiv \int U(g)_{a_1b_1}U(g)_{a_2b_2}U(g)_{a_3b_3}d\mu(g) \equiv \int g_{a_1b_1}g_{a_2b_2}g_{a_3b_3}d\mu(g) = \frac{1}{6}\,\epsilon_{a_1a_2a_3}\epsilon_{b_1b_2b_3}\,, &\text{(B2)}\\
\mathcal{I}_4 &\equiv \int g_{a_1b_1}g^{-1}_{a_2b_2}g_{a_3b_3}g^{-1}_{a_4b_4}\,d\mu(g)\\
&= \frac{1}{8}\,[\delta_{a_1b_2}\delta_{a_3b_4}\delta_{b_1a_2}\,\delta_{b_3a_4} + (a_2 \rightleftarrows a_4, b_2 \rightleftarrows b_4)] - \frac{1}{24}\,[\delta_{a_1b_2}\delta_{a_3b_4}\delta_{b_1a_4}\,\delta_{b_3a_2} + (a_2 \rightleftarrows a_4, b_2 \rightleftarrows b_4)] &\text{(B3)}\\
\mathcal{I}_6 &\equiv \int g_{a_1b_1}g_{a_2b_2}g_{a_3b_3}g_{a_4b_4}g_{a_5b_5}g_{a_6b_6}\,d\mu(g)\\
&= \frac{2}{3!4!}\,[\epsilon_{a_1a_2a_3}\epsilon_{b_1b_2b_3}\epsilon_{a_4a_5a_6}\epsilon_{b_4b_5b_6} + \epsilon_{a_1a_2a_4}\epsilon_{b_1b_2b_4}\epsilon_{a_3a_5a_6}\epsilon_{b_3b_5b_6}+\\
&\quad \epsilon_{a_1a_2a_5}\epsilon_{b_1b_2b_5}\epsilon_{a_3a_4a_6}\epsilon_{b_3b_4b_6} + \epsilon_{a_1a_2a_6}\epsilon_{b_1b_2b_6}\epsilon_{a_3a_4a_5}\epsilon_{b_3b_3b_5}+\\
&\quad \epsilon_{a_1a_3a_4}\epsilon_{b_1b_3b_3}\epsilon_{a_2a_5a_6}\epsilon_{b_2b_5b_6} + \epsilon_{a_1a_3a_5}\epsilon_{b_1b_3b_5}\epsilon_{a_2a_4a_6}\epsilon_{b_2b_4b_6}+\\
&\quad \epsilon_{a_1a_3a_6}\epsilon_{b_1b_3b_6}\epsilon_{a_2a_4a_5}\epsilon_{b_2b_4b_5} + \epsilon_{a_1a_4a_5}\epsilon_{b_1b_4b_5}\epsilon_{a_2a_3a_6}\epsilon_{b_2b_3b_6}+\\
&\quad \epsilon_{a_1a_4a_6}\epsilon_{b_1b_4b_6}\epsilon_{a_2a_3a_5}\epsilon_{b_2b_3b_5} + \epsilon_{a_1a_5a_6}\epsilon_{b_1b_5b_6}\epsilon_{a_2a_3a_4}\epsilon_{b_2b_3b_4}]\,. &\text{(B4)}
\end{aligned}
$$

Only the integral $\mathcal{I}_3$ occurs in self-avoiding paths. In addition to the self-avoiding paths above there are also disjoint boxes. Boxes are described by unit squares where one side has three temporal bonds of the same orientation and each of the other three sides has two bonds of opposite orientation associated with the integral $\mathcal{I}_2$. Boxes may also have stems which extend the temporal bonds from the bottom or the top of the square.

We first consider the simplest case of a baryon field with total isospin 3/2. By flavor symmetry, it is enough to calculate the behavior for one of the members of the decuplet. For simplicity, we do this for the Δ^{++} particle in Eq. (28). Since all isospin indices are equal, we simply suppress them here, and denote the s spin state of this baryon field by ϕ_s.

For the self-avoiding paths contributing to G_d with fixed maximal total isospin, after performing the gauge integrals and integrating over the intermediate fermions (Wick's theorem), we obtain the following general formula for the contribution of path p and length L

$$
\langle \phi_s(0)\bar{\phi}_s(x)\rangle \;:\; (-1)^{L+1}\left(\frac{\kappa}{2}\right)^{3L} \begin{cases} [\Gamma^p_{33}]^3 & ,\quad s=3/2\\ [\Gamma^p_{33}]^2\,\Gamma^p_{44} + 2\Gamma^p_{33}\Gamma^p_{34}\Gamma^p_{43} & ,\quad s=1/2\,,\end{cases} \tag{B5}
$$

where L is the path length, i.e. the number of lattice bonds in the path and Γ^p_{ij} is the ij element of the ordered product of spin $\Gamma^{\epsilon e^\mu}$ matrices along the path from 0 to x. For instance, if $x = e^0 + e^1 + e^2$, and if the chosen path goes like $0 \to e^2 \to e^1 + e^2 \to e^0 + e^1 + e^2$, then $\Gamma^p = \Gamma^{e^2}\Gamma^{e^1}\Gamma^{e^0}$. In general, if the path is specified by $0 \to x_1 \to x_2 \ldots \to x_n \to x$, then $\Gamma^p \equiv \Gamma^{0\to x} = \Gamma^{x_1}\Gamma^{x_2-x_1}\ldots\Gamma^{x-x_n}$, and $L = n+1$. The values for the other isospins are obtained by application of the isospin lowering operator. Furthermore, we observe that a similar formula holds also for nondiagonal elements $\langle \phi_s(0)\bar{\phi}_{s'\neq s}(x)\rangle$.

We give a brief deduction of the path formula of Eq. (B5). Expanding the exponential of the action in the numerator of $\langle \phi_s(0)\bar{\phi}_s(x)\rangle$, we pick up three hopping terms for each bond of the path. Using $\mathcal{I}_3$, and carrying out the Fermi integration over intermediate fields, we obtain

$$
(-1)^{L+1}\left(\frac{\kappa}{2}\right)^{3L}[\mathrm{perm}(\delta_{\vec{s}\vec{\alpha}})/n_s]\;[\mathrm{perm}(\delta_{\vec{\beta}\vec{s}'})/n_{s'}]\,\Gamma^p_{\alpha_1\beta_1}\Gamma^p_{\alpha_3\beta_2}\Gamma^p_{\alpha_3\beta_3}\,. \tag{B6}
$$

Here $\vec{s} = 333$ for $s = 3/2$ (all spins of individual quarks are up!) and $\vec{s} = 334$ for $s = 3/2$ (two spins of individual quarks are up and one is down!), $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$ are collective lower indices and take the values 3 or 4. Arising from normalizations, we also have $n_s = 6$ for $s = 3/2$ and $n_s = 2\sqrt{3}$ for $s = 1/2$. Next, $\delta_{\vec{s}\vec{\alpha}} = [\delta_{s_i\alpha_j}]^3_{i,j=1}$ and perm means the permanent (the same sum of terms as the determinant but with only plus signs) and arises from the usual determinant emerging from the fermionic integral times the sign coming from the Levi-Civita's in $\mathcal{I}_3$ and in the ϕ's. Finally, calculating the permanents gives the result.

We will also need to use a type of formula as in Eq. (B5) for the proton-proton correlation, which is obtained in a similar way. The analogue of Eq. (B5) here is given by the formula

$$
\langle p_+(0)\bar{p}_+(x)\rangle \;:\; (-1)^{L+1}\left(\frac{\kappa}{2}\right)^{3L}\left([\Gamma^p_{33}]^2\,\Gamma^p_{44} - \Gamma^p_{33}\Gamma^p_{34}\Gamma^p_{43}\right)\,. \tag{B7}
$$

It is derived as above but with the permanents in brackets in Eq. (B6) replaced by $\{[\mathrm{perm}(\delta_{\vec{t}\vec{\alpha}}\delta_{\vec{h}\vec{f}})/n_p] - [\mathrm{perm}(\delta_{\vec{t}\vec{\alpha}}\delta_{\vec{h}'\vec{f}})/n_p]\}$, with $n_p = 3\sqrt{2}$ and where $[\delta_{\vec{t}\vec{\alpha}}\delta_{\vec{h}\vec{f}}]_{ij} \equiv \delta_{(\vec{t})^i(\vec{\alpha})^j}\delta_{(\vec{h})^i(\vec{f})^j}$, $\vec{t} = (+,-,+)$ lies in the lower component spin space, $\vec{h} = (u,d,u)$ and $\vec{h}' = (d,u,u)$ lie in the flavor space.

In order to simplify the proof of Theorem 4 , we give the following two Lemmas:

Lemma B2 *The following relations are satisfied by G and Γ*

1. $G_{ss'}(x) = [G_{s's}(x)]^*$;
2. $G_{ss'}(-x^0, \vec{x}) = [G_{s's}(x)]^*$;
3. $G_{ss}(x) = G_{ss}(-x^0, \vec{x}) = [G_{ss}(x^0, \vec{x})]^*$;

and the same for $\Gamma_{ss'}(x)$.

Proof: The Lemma follows using time reversal and parity. ■

Using this Lemma, we only have to consider $x^0 > 0$ in the proof of Theorem 4 . In a similar way, the following Lemma exploits symmetries to relate the value of G_{ss} at different points.

Lemma B3 *For $\alpha, \beta = 0, 1$ and any $\epsilon, \epsilon' \in \{-1, +1\}$, the following relations are verified:*

1. $\langle \phi_s(0)\bar{\phi}_s(\alpha e^0 + \epsilon e^i + \beta\epsilon' e^j)\rangle = \langle \phi_s(0)\bar{\phi}_s(\alpha e^0 + e^i + \beta e^j)\rangle$;
2. $\langle \phi_s(0)\bar{\phi}_s(\alpha e^0 + e^1 + \beta e^3)\rangle = \langle \phi_s(0)\bar{\phi}_s(\alpha e^0 + e^2 + \beta e^3)\rangle$.

Proof: *First item:* Consider $\alpha = 0, 1$, $\beta = 1$. For $ij = 12$, use $\pi/2$ rotations about e^3. For $ij = 13$, use parity and e^3 reflections. For $ij = 23$, use $\pi/2$ rotations about e^3 on the $ij = 13$ value. For $\beta = 0$, and $ij = 12$, use parity. *Second item:* Use $\pi/2$ rotations about e^3 on the $ij = 13$ value. ■

Using these two Lemmas, we only need to prove Theorem 4 for $\epsilon, \epsilon', \epsilon'' = 1$, for $ij = 12, 13$. We also note that many possible configurations are shown to be zero using the *come and go* property of the Γ's, ($\mu, \nu = 0, 1, 2, 3$, and $\epsilon = \pm 1$)

$$\Gamma^{\epsilon e^\mu}\,\Gamma^{-\epsilon e^\mu} \;=\; 0\,. \tag{B8}$$

For example, if a path doubles back upon itself at an isolated point then at this point the come and go property of Eq. (B8) holds to give a zero contribution. Also, possible contributions can give zero due to an imbalance of the number of fermions or the number of fermion components at a site.

Also, other useful properties involving Γ matrices ($\mu, \nu = 0, 1, 2, 3$, and ϵ, $\epsilon' = \pm 1$) used in evaluating possible contributions are

$$\begin{aligned}\Gamma^{\epsilon e^\mu}\,\Gamma^{\epsilon e^\mu} &= -2\Gamma^{\epsilon e^\mu}\,, &\text{(B9)}\\ \Gamma^{\epsilon e^\mu}\,\Gamma^{\epsilon' e^\nu} &= 2I_4 - \Gamma^{-\epsilon' e^\nu}\,\Gamma^{-\epsilon e^\mu}\,. &\text{(B10)}\end{aligned}$$

Especially, the property in Eq. (B9) shows that lattice bond segments in a straight line behave similarly. Finally, the property of Eq. (B10) is useful to sum over different orders in a path with fixed endpoints.

Proof of Theorem 4 : All the c constants to which we refer here are the ones appearing in Eqs. (31) and (32). To compute the contributions to G and Γ that we need to establish Theorem 4 , we use the above results on gauge integrals [see Eqs. (B1)-(B4)].

Again, we first consider the decuplet case.

- We begin by considering $x = 0$. Using the symmetry of $\pi/2$ rotations about e^3 shows that the off-diagonal elements are zero. The gauge integral properties and the Clebsch-Gordan properties show that the first nonvanishing contribution occurs at κ^8, and consists of two paths which go around a square in opposite directions. The square has one vertex at zero. A detailed calculation shows that c_8 is independent of the spin s.

- Take now $x = e^0$. The κ^3 contribution is a direct application of the path formula of Eq. (B5). The κ^9 contributions come from paths of the type 0, e^j, $e^0 + e^j$, e^0, which we call U's, and boxes with three positively oriented bonds between 0 and e^0, and oppositely oriented bonds on the other three sides of the square with vertices 0, e^j, $e^0 + e^j$, e^0. A detailed calculation (see below) shows that the contributions are diagonal and independent of the spin s.

- For $x = e^j$, the κ^3 contributions are obtained from the path formula of Eq. (B5) and the κ^9 contributions come from boxes and U's.

- For $x = e^\mu + e^\nu$, we make a straightforward application of the path formula. The same holds also for $x = e^0 + 2e^j$, $x = 2e^0 + e^j$, $x = e^0 + e^1 + e^2$, $x = e^0 + e^1 + e^3$, $x = 3e^0$ and $x = 4e^0$. There are vertical (temporally oriented) U and box contributions. The vertical contributions are given by the path formula as well as the U contributions. The box contributions are calculated as for $x = e^0$. Possible contributions which are vertical backtracking paths, i.e. the path 0, e^0, $2e^0$, e^0, $2e^0$, $3e^0$, $4e^0$ are zero using imbalance of fermion components at e^0 or at $2e^0$.

The computation of all the above contributions to G is a direct application of the path formula of Eq. (B5), except for the box contribution.

For the octet, the computation of all the non-box contributions above are similar. From flavor symmetry, it is enough to consider the proton-proton contributions with both spins $J_z = +1/2$. The path formula of Eq. (B7) is used again. An important remark is that the first terms in Eqs. (B5) and (B7) end up to be the same. That this is true follows from the fact that Γ^p is multiple of the identity in the lower spin sector, which is seen using Eqs. (B8)-(B9) and from $(\Gamma^j\Gamma^{\pm 0}\Gamma^{-j})_{\ell ower} = \mp 2I_2$. For the same reason, the second terms do not contribute. For the points $x = r\epsilon e^\mu$, $r = 1, 2, 3, 4$, $\epsilon e^0 + \epsilon' e^j$, $\epsilon e^0 + 2\epsilon' e^j$, $2\epsilon e^0 + \epsilon' e^j$, $3\epsilon e^0 + \epsilon' e^j$, $4\epsilon e^0 + \epsilon' e^j$, the results for G in the proton-proton case and the decuplet are then the same to the stated κ order. Since these are the only contributions that enter to define Γ up to the orders stated in Theorem 4, the results for Γ are also the same for these points. We are then left to prove the short distance behavior of the proton-proton G and Γ for the points $x = \epsilon e^i + \epsilon' e^j$, $x = \epsilon e^0 + \epsilon' e^i + \epsilon'' e^j$, for $ij = 12, 13, 23$, which follow from the path formula Eq. (B7).

In both the octet and the decuplet cases, the contribution of a box is the most subtle to compute, and we give a detailed treatment.

Considering the decuplet case and taking the maximum isospin state Δ^{++}, the box with vertices at 0, e^j, $e^0 + e^j$, e^0, and for $x = e^0$, after carrying out the gauge integrations and the intermediate Fermi integrations at e^j and $e^0 + e^j$, up to a factor $c\kappa^9$, taking, as usually, summation over repeated indices and using the $\hat{b}$ fields of Eq. (1), we get

$$\langle \phi_s \bar{b}_{\vec{\alpha}\vec{f}} \bar{\psi}_{a\alpha_1 f_1} \psi_{a\beta_2 f_2} \rangle^{(0)} \, (\Gamma^j \Gamma^0 \Gamma^{-j})_{\alpha_1\beta_1} (\Gamma^j \Gamma^{-0} \Gamma^{-j})_{\alpha_2\beta_2} \langle \bar{\psi}_{b\alpha_2 f_2} \psi_{b\beta_1 f_1} b_{\vec{\alpha}\vec{f}} \bar{\phi}_{s'} \rangle^{(0)} ,$$

where the γ's are lower components, and we use the notation $\Gamma^{\pm e^\mu} \equiv \Gamma^{\pm\mu}$. If the $\bar{\psi}_a$ and the ψ_a contract among themselves, the come and go property gives zero. Thus the $\psi_{a\beta_2}$ only contracts with the $\bar{\psi}_{c_k\gamma_k}$'s and, similarly, the $\psi_{b\beta_2}$ only contracts with the $\bar{\psi}_{d_\ell\gamma_\ell}$'s. Furthermore, as $\bar{\psi}_{a\alpha_1}$ contracts with ϕ_{s_1}, and $\psi_{b\beta_1}$ contracts with ϕ_{s_2}, the spin indices α_1, β_2, α_2, β_1 are all lower and we have $(\Gamma^j\Gamma^{\pm 0}\Gamma^{-j})_{\ell ower} = \mp 2I_2$. Thus, up to a factor, we can write the above as, with $\phi_s \to \phi_{\vec{s}\vec{k}}$ and $\phi_{s'} \to \phi_{\vec{s}'\vec{k}'}$,

$$\langle \phi_{\vec{s}\vec{k}} \bar{b}_{\vec{\alpha}\vec{f}} \bar{\psi}_{a\alpha h_1} \psi_{a\beta h_2} \rangle^{(0)} \, \langle \bar{\psi}_{b\beta h_2} \psi_{b\alpha h_1} b_{\vec{\alpha}\vec{f}} \bar{\phi}_{\vec{s}'\vec{k}'} \rangle^{(0)} .$$

We note that for the present case $\phi_{\vec{s}\vec{k}} = b_{\vec{s}\vec{k}}/n_s$ and $\phi_{\vec{s}'\vec{k}'} = b_{\vec{s}'\vec{k}'}/n_{s'}$.

Using conjugation symmetry ($\psi \leftrightarrow \bar{\psi}$), which holds for $\kappa = 0$, the second average is equal to the first with $\vec{s}\vec{k} \to \vec{s}'\vec{k}'$. To evaluate the first average, move the last ψ to the first place and use the Laplace expansion to right the first factor as

$$F = F_1 + F_2 + F_3 ,$$

where

$$F_i = -\frac{6}{n_s} \, \delta_{\beta\gamma_i} \delta_{h_2 f_i} \, \mathrm{perm}(\delta_{\vec{s}\vec{\sigma}^i} \delta_{\vec{k}\vec{\tau}^i}) \quad , \quad i = 1, 2, 3 ,$$

with $\vec{\sigma}^1 = (\gamma_2, \gamma_3, \alpha)$, $\vec{\tau}^1 = (f_2, f_3, h_1)$; $\vec{\sigma}^2 = (\gamma_1, \gamma_3, \alpha)$, $\vec{\tau}^2 = (f_1, f_3, h_1)$; $\vec{\sigma}^3 = (\gamma_1, \gamma_2, \alpha)$, $\vec{\tau}^3 = (f_1, f_2, h_1)$. We write and evaluate the second average in a similar manner.

We now evaluate the box contribution FF for the cases $J_z = 3/2, 1/2$, $s = s'$ in the state of maximum $I_3 = 3/2$. We have to perform the contractions of indices of a product of permanents in FF, such as for F_iF_j. We use the easily shown result that if the 6 terms of the permanent are all equal, then the $i = j$ permanent-permanent contraction is 36; if there are 3 distinct pairs the result is 12 and if the 6 terms are distinct the result is 6. Also, for F_iF_j, $i \neq j$, due to the prefactor delta functions, $F_iF_j = F_iF_i/6$ (no i sum).

We find

- For $J_z = 3/2$, we have $\vec{s} = (+, +, +)$, $\vec{k} = (u, u, u) \equiv (1, 1, 1)$ and $n_s = 6$. Then, $F_iF_i = 6^3$, $i = 1, 2, 3$, $F_iF_{j\neq i} = 6^2$ and $FF = 3 \times 6^3 + 6 \times 6^2 = 6^4(2/3)$;

- $J_z = 1/2$, we have $\vec{s} = (+, +, -)$, $\vec{k} = (u, u, u) \equiv (1, 1, 1)$ and $n_s = 2\sqrt{3}$. Then, $F_iF_i = 6^3$, $i = 1, 2, 3$, $F_iF_{j\neq i} = 6^2$ and $FF = 3 \times 6^3 + 6 \times 6^2 = 6^4(2/3)$.

For the octet case, it is enough to evaluate the proton-proton contribution p_+p_+. Here, the ϕ's are linear combinations of two b's, $F_i \to F_i^1 - F_i^2$, $n_s \to n_p = 3\sqrt{2}$, and in F_i^1, $\vec{s} = (+,-,+)$, $\vec{k}^1 = (u,d,u) \equiv (1,2,1)$; in F_i^2, $\vec{s} = (+,-,+)$, $\vec{k}^1 = (d,u,u) \equiv (2,1,1)$. We find $F_i^1 F_j^2 = 0$, $i,j = 1, 2, 3$; $F_i^1 F_i^1 = 2F_i^2 F_i^2 = \frac{2\times 6^4}{2\times 9}$; $F_i^2 F_j^2 = \frac{6^3}{2\times 9}$, $i \neq j$. With this, $FF = \frac{1}{2\times 9}[3(2\times 6^4 + 6^4) + 6(12\times 6^2 + 6^3)] = 6^4(2/3)$.

Thus, the spin $J_z = 3/2, 1/2$ and the proton box contributions are all equal. For a box with r stems, as described above, we have, up to a sign, an additional multiplicative factor κ^{3r}.

The proof of Theorem 4 is then complete. ■

APPENDIX C: Particle Basis and Gell'Mann-Ne'eman Non-Dynamical Particle Construction Based on the Flavor Group $\mathrm{SU}(3)_f$

Here we recall the non-dynamical flavor group construction of the eightfold way, and establish its relation to the particle basis of Section II C.

The basic quantity we start with is the decomposition of the tensor product representation of $\mathrm{SU}(3)_f$

$$3 \times 3 \times 3 = 10 \oplus 8 \oplus 8 \oplus 1\,.$$

The use of isospin and hypercharge raising and lowering operators to separately move from state to state within the octet and the decuplet is discussed in Appendix A. Also, the eigenvalues of the third component of total isospin and hypercharge are indicated in Figures 1 and 2 of Section II C.

We first deal with the spin 1/2 baryons appearing in the above eight dimensional representation giving the octet. Using also the Young tableaux terminology (see e.g. Ref. [46]), the following eight states are orthogonal in the total isospin, third-component of isospin, and in hypercharge Y or in the strangeness S (with a normalization such that $\langle\cdot\rangle^{(0)} = -1$).

OCTET:

- uud : $p_\pm = \frac{1}{3\sqrt{2}}\epsilon_{abc}[\bar\psi_{a+u}\bar\psi_{b-d} - \bar\psi_{a+d}\bar\psi_{b-u}]\bar\psi_{c\pm u}$, $\begin{array}{|c|c|}\hline u & u \\ \hline d & \\ \cline{1-1}\end{array}$, $I = 1/2$;
- udd : $n_\pm = \frac{1}{3\sqrt{2}}\epsilon_{abc}[\bar\psi_{a+u}\bar\psi_{b-d} - \bar\psi_{a+d}\bar\psi_{b-u}]\bar\psi_{c\pm d}$, $\begin{array}{|c|c|}\hline u & d \\ \hline d & \\ \cline{1-1}\end{array}$, $I = 1/2$;
- uss : $\Xi^0_\pm = \frac{1}{3\sqrt{2}}\epsilon_{abc}[\bar\psi_{a+u}\bar\psi_{b-s} - \bar\psi_{a+s}\bar\psi_{b-u}]\bar\psi_{c\pm s}$, $\begin{array}{|c|c|}\hline u & s \\ \hline s & \\ \cline{1-1}\end{array}$, $I = 1/2$;
- dss : $\Xi^-_\pm = \frac{1}{3\sqrt{2}}\epsilon_{abc}[\bar\psi_{a+d}\bar\psi_{b-s} - \bar\psi_{a+s}\bar\psi_{b-d}]\bar\psi_{c\pm s}$, $\begin{array}{|c|c|}\hline d & s \\ \hline s & \\ \cline{1-1}\end{array}$, $I = 1/2$;
- uus : $\Sigma^+_\pm = \frac{1}{3\sqrt{2}}\epsilon_{abc}[\bar\psi_{a+u}\bar\psi_{b-s} - \bar\psi_{a+s}\bar\psi_{b-u}]\bar\psi_{c\pm u}$, $\begin{array}{|c|c|}\hline u & u \\ \hline s & \\ \cline{1-1}\end{array}$, $I = 1$;
- $\Sigma^0_\pm = \frac{\epsilon_{abc}}{6}\,(2\bar\psi_{a\pm u}\bar\psi_{b\pm d}\bar\psi_{c\mp s} - \bar\psi_{a-u}\bar\psi_{b+d}\bar\psi_{c\pm s} - \bar\psi_{a+u}\bar\psi_{b-d}\bar\psi_{c\pm s})$, $I = 1$;
- dds : $\Sigma^-_\pm = \frac{1}{3\sqrt{2}}\epsilon_{abc}[\bar\psi_{a+d}\bar\psi_{b-s} - \bar\psi_{a+s}\bar\psi_{b-d}]\bar\psi_{c\pm d}$, $\begin{array}{|c|c|}\hline d & d \\ \hline s & \\ \cline{1-1}\end{array}$, $I = 1$;
- usd : $\Lambda_\pm = \frac{1}{2\sqrt{3}}\epsilon_{abc}[\bar\psi_{a+u}\bar\psi_{b-d} - \bar\psi_{a+d}\bar\psi_{b-u}]\bar\psi_{c\pm s}$, $\begin{array}{|c|c|}\hline u & s \\ \hline d & \\ \cline{1-1}\end{array}$, $I = 0$.

We note that, associating F with $\begin{array}{|c|c|}\hline u & d \\ \hline s & \\ \cline{1-1}\end{array}$, and letting the superscript n denote the fields with pre-factor 1, then $\Sigma^{0,n} = 2F^n - \Lambda^n$, $F^n = \epsilon_{abc}[\psi_{a+u}\psi_{b+d}\psi_{c-s} - \psi_{a+s}\psi_{b+d}\psi_{c-u}]$.

For the ten-dimensional representation giving the decuplet, with the same normalization condition as above, we obtain:

DECUPLET:

- uuu : $\Delta^{++}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{6}\,\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\pm u}$, $\Delta^{++}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\,\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\mp u}$, $\begin{array}{|c|c|c|}\hline u & u & u \\ \hline\end{array}$, $I = 3/2$;
- uud : $\Delta^{+}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{6}(\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\mp d} + 2\bar\psi_{a\pm u}\bar\psi_{b\mp u}\bar\psi_{c\pm d})$, $\Delta^{+}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\,\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\pm d}$, $\begin{array}{|c|c|c|}\hline u & u & d \\ \hline\end{array}$, $I = 3/2$;

- udd : $\Delta^0_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{6}(2\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm d}\bar{\psi}_{c\mp d}+\bar{\psi}_{a\mp u}\bar{\psi}_{b\pm d}\bar{\psi}_{c\pm d})$, $\Delta^0_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\,\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm d}\bar{\psi}_{c\pm d}$, $\boxed{u\,|\,d\,|\,d}$, $I = 3/2$;
- ddd : $\Delta^-_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\,\bar{\psi}_{a\pm d}\bar{\psi}_{b\pm d}\bar{\psi}_{c\mp d}$, $\Delta^-_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{6}\,\bar{\psi}_{a\pm d}\bar{\psi}_{b\pm d}\bar{\psi}_{c\pm d}$, $\boxed{d\,|\,d\,|\,d}$, $I = 3/2$;
- uus : $\Sigma^{*+}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm u}\bar{\psi}_{c\pm s}$, $\Sigma^{*+}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{6}\,(\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm u}\bar{\psi}_{c\mp s}+2\bar{\psi}_{a\pm u}\bar{\psi}_{b\mp u}\bar{\psi}_{c\pm s})$, $\boxed{u\,|\,u\,|\,s}$, $I = 1$;
- uds : $\Sigma^{*0}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{6}\,\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm d}\bar{\psi}_{c\pm s}$, $\Sigma^{*0}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{3\sqrt{2}}(\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm d}\bar{\psi}_{c\mp s} + \bar{\psi}_{a\pm u}\bar{\psi}_{b\mp d}\bar{\psi}_{c\pm s} + \bar{\psi}_{a\mp u}\bar{\psi}_{b\pm d}\bar{\psi}_{c\pm s})$, $\boxed{u\,|\,d\,|\,s}$, $I = 1$;
- dds : $\Sigma^{*-}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar{\psi}_{a\pm d}\bar{\psi}_{b\pm d}\bar{\psi}_{c\pm s}$, $\Sigma^{*-}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{6}\,(\bar{\psi}_{a\pm d}\bar{\psi}_{b\pm d}\bar{\psi}_{c\mp s}+2\bar{\psi}_{a\pm d}\bar{\psi}_{b\mp d}\bar{\psi}_{c\pm s})$, $\boxed{d\,|\,d\,|\,s}$, $I = 1$;
- uss : $\Xi^{*0}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm s}\bar{\psi}_{c\pm s}$, $\Xi^{*0}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{6}\,(\bar{\psi}_{a\mp u}\bar{\psi}_{b\pm s}+2\bar{\psi}_{a\pm u}\bar{\psi}_{b\mp s})\bar{\psi}_{c\pm s}$, $\boxed{u\,|\,s\,|\,s}$, $I = 1/2$.
- dss : $\Xi^{*-}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar{\psi}_{a\pm d}\bar{\psi}_{b\pm s}\bar{\psi}_{c\pm s}$, $\Xi^{*-}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{6}\,(\bar{\psi}_{a\mp d}\bar{\psi}_{b\pm s}+2\bar{\psi}_{a\pm d}\bar{\psi}_{b\mp s})\bar{\psi}_{c\pm s}$, $\boxed{d\,|\,s\,|\,s}$, $I = 1/2$;
- sss : $\Omega^-_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{6}\bar{\psi}_{a\pm s}\bar{\psi}_{b\pm s}\bar{\psi}_{c\pm s}$, $\Omega^-_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar{\psi}_{a\pm s}\bar{\psi}_{b\pm s}\bar{\psi}_{c\mp s}$, $\boxed{s\,|\,s\,|\,s}$, $I = 0$.

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[51] Note that in the basis where $\tilde{\Gamma}$ is triangular, its eigenvalues are the diagonal elements, and the same holds for the cofactor matrix $\mathrm{cof}[\tilde{\Gamma}]$ which has one dimension less than $\tilde{\Gamma}$. Thus, in the quotient of Eq. (20), all the zeroes of $\det\tilde{\Gamma}$ are singularities of $\tilde{G} \equiv \tilde{\Gamma}^{-1}$.
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