Dynamical Eightfold Way Mesons in Strongly Coupled Lattice QCD

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(Dated: November 9, 2007)

We consider a $3+1$ lattice QCD model with three quark flavors, local $\mathrm{SU}(3)_c$ gauge symmetry, global $\mathrm{SU}(3)_f$ isospin or flavor symmetry, in an imaginary-time formulation and with strong coupling (a small hopping parameter $\kappa > 0$ and a plaquette coupling $\beta > 0$, $0 < \beta \ll \kappa \ll 1$). Associated with the model there is an underlying physical quantum mechanical Hilbert space $\mathcal{H}$ which, via a Feynman-Kac formula, enables us to introduce spectral representations for correlations and obtain the low-lying energy-momentum spectrum exactly. Using the decoupling of hyperplane method and concentrating on the subspace $\mathcal{H}_e \subset \mathcal{H}$ of vectors with an *even* number of quarks, we obtain the one-particle spectrum showing the existence of 36 meson states from dynamical first principles, i.e. directly from the quark-gluon dynamics. Besides the $\mathrm{SU}(3)_f$ quantum numbers (total hypercharge, quadratic Casimir C_2, total isospin and its 3rd component), the basic excitations also carry spin labels. The total spin operator J and its z-component J_z are defined using $\pi/2$ rotations about the spatial coordinate axes and agree with the infinitesimal generators of the continuum for improper zero-momentum meson states. The eightfold way meson particles are given by linear combinations of these 36 states and can be grouped into three $\mathrm{SU}(3)_f$ nonets associated with the vector mesons ($J = 1,\ J_z = 0, \pm 1$) and one nonet associated with the pseudo-scalar mesons ($J = 0$). Each nonet admits a further decomposition into a $\mathrm{SU}(3)_f$ singlet ($C_2 = 0$) and octet ($C_2 = 3$). The particles are detected by isolated dispersion curves $w(\vec{p})$ in the energy-momentum spectrum. They are all of the form, for $\beta = 0$, $w(\vec{p}) = -2\ln\kappa - 3\kappa^2/2 + (1/4)\kappa^2 \sum_{j=1}^{3} 2(1-\cos p^j) + \kappa^4 r(\kappa,\vec{p})$, with $|r(\kappa,\vec{p})| \leq \text{const}$. For the pseudo-scalar mesons $r(\kappa,\vec{p})$ is jointly analytic in κ and p^j, for $|\kappa|$ and $|\mathrm{Im}\, p^j|$ small. The meson masses are given by $m(\kappa) = -2\ln\kappa - 3\kappa^2/2 + \kappa^4 r(\kappa)$, with $r(0) \neq 0$ and $r(\kappa)$ real analytic; they are also analytic in β. For a fixed nonet, the mass of the vector mesons are independent of J_z and are all equal within each octet. All singlet masses are also equal for the vector mesons. For $\beta = 0$, up to and including $\mathcal{O}(\kappa^4)$, for each nonet, the masses of the octet and the singlet are found to be equal. All members of each octet have identical dispersions. Other dispersion curves may differ. Indeed, there is a pseudo-scalar, vector meson mass splitting (between $J = 0$ and $J = 1$) given by $2\kappa^4 + \mathcal{O}(\kappa^6)$; at $\beta = 0$, analytic in β and the splitting persists for $\beta << \kappa$. Using a correlation subtraction method, we show the 36 meson states give the only spectrum in $\mathcal{H}_e$ up to near the two-meson threshold of $\approx -4\ln\kappa$. Combining our present result with a similar one for baryons (of asymptotic mass $-3\ln\kappa$) shows that the model does exhibit confinement up to near the two-meson threshold.

PACS numbers: 11.15.Ha, 02.30.Tb, 11.10.St, 24.85.+p
Keywords: Lattice QCD, Excitation Spectrum, Eightfold Way, Spectral Analysis

I. INTRODUCTION

A fundamental classification scheme in elementary particle physics is the eightfold way group theoretical construction based on the global quark flavor or isospin symmetry $\mathrm{SU}(3)_f$ by Gell'Mann-Ne'eman (see Refs. [1, 2]). This scheme is expected to be incorporated in the dynamical $\mathrm{SU}(3)_c$ local gauge QCD model describing the interaction of quarks and gluons. To understand the low-lying energy-momentum (E-M) spectrum and confinement in QCD is a challenging problem. In Ref. [3], these problems were approached by adopting a lattice approximation in an imaginary-time functional integral formulation. The use of the lattice in different contexts turned out to be very fruitful. For example, the strong coupling expansion was employed to determine the particle content of the model and to give answers to questions not attainable using perturbation theory (see Refs. [3–5]). A mathematically rigorous treatment in an imaginary-time lattice formulation was devised in Ref. [6] where the quantum mechanical physical Hilbert space $\mathcal{H}$ and E-M operators were constructed. A Feynman-Kac (F-K) formula was also established.

The low-lying E-M spectrum (one-particle and two-particle bound states) was rigorously determined exactly in Refs. [7, 8] for increasingly complex $\mathrm{SU}(3)_c$ lattice QCD models with one and two flavors in the strong cou-

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pling regime, i.e., we consider the model in the region of small hopping parameter κ and much smaller plaquette parameter β. In this regime, more recently, the $\mathrm{SU}(3)_f$ scheme was validated in Refs. [9, 10] by obtaining the spectrum of all the 56 gauge-invariant eightfold way baryons (of asymptotic mass $-3\ln\kappa$), and their antiparticles exactly, from the quark-gluon dynamics in $3+1$ dimensional $\mathrm{SU}(3)_c$ lattice QCD with three flavors. These baryons can be grouped into four decuplets of total spin $J=3/2$, $J_z=\pm 3/2, \pm 1/2$, and two octets ($J=1/2$, $J_z=\pm 1/2$). Considering the subspace $\mathcal{H}_o \subset \mathcal{H}$ of vectors with an *odd* number of quark fields the eightfold way baryon and anti-baryon spectrum is shown to be the only spectrum up to near the meson-baryon energy threshold ($\approx -5\ln\kappa$). Hence, confinement is proven in $\mathcal{H}_o$. The baryon masses are given by convergent expansions in the coupling parameters. The reason for the restriction $\beta << \kappa$ is that in this region of parameters the hadron spectrum is the low-lying spectrum. If, on the other hand, $\beta >> \kappa$ then the low-lying spectrum consists of only glueballs and their excitations (see Ref. [11]).

Starting from a F-K formula, these results are obtained from spectral representations for the two-baryon functions. We emphasize that claiming spectral results based solely on the behavior of a correlation function or its Fourier transform unfortunately tells us nothing about the energy-momentum spectrum of the associated quantum mechanical version of model unless a connection is established between the correlation functions and the energy-momentum operators of the associated model. It is unfortunate that this basic fact is not always taken into account! Moreover, we point out that only establishing exponential decay of correlations, in principle, says nothing about the spectrum and, even in cases where the associated mass turns out to be a spectral point, this procedure says nothing about what happens in the spectrum above this mass. This is not enough to show the upper gap property and to ensure that the dispersion curves defining the particles are isolated.

We remark that earlier works (see Refs. [12, 13]) devoted to the determination of the low-lying energy-momentum spectrum in lattice QCD, via the zeros of uncontrolled expansions in the denominator of the Fourier transform of approximate propagators, leaves unanswered the question of the nature and the existence of the supposed singularity which is identified with the mass.

Here, we complete the exact determination of the one-particle E-M spectrum by considering the *even* subspace $\mathcal{H}_e$ and show the existence of the eightfold way mesons (of asymptotic mass $-2\ln\kappa$). The three-flavor model with global $\mathrm{SU}(3)_f$ and local $\mathrm{SU}(3)_c$ gauge symmetry we consider is defined by the Wilson action

$$\begin{aligned} S \;=\; & \tfrac{\kappa}{2}\sum \bar\psi_{a,\alpha,f}(u)\Gamma^{\sigma e^\mu}_{\alpha\beta}(g_{u,u+\sigma e^\mu})_{ab}\psi_{b,\beta,f}(u+\sigma e^\mu) \\ & + \textstyle\sum_{u\in\mathbb{Z}^4_o}\bar\psi_{a,\alpha,f}(u)M_{\alpha\beta}\psi_{a,\beta,f}(u) - \frac{1}{g_0^2}\sum_p \chi(g_p)\,. \end{aligned} \tag{1}$$

The partition function is $Z=\int e^{-S(\psi,\bar\psi,g)}\,d\psi\,d\bar\psi\,d\mu(g)$, and for $F(\bar\psi,\psi,g)$, the normalized correlations are denoted by $\langle F\rangle = \frac{1}{Z}\int F(\bar\psi,\psi,g)\,e^{-S(\psi,\bar\psi,g)}\,d\psi\,d\bar\psi\,d\mu(g)$. In the strong coupling regime, we take the hopping parameter $\kappa>0$ to be small but much larger than the plaquette coupling $\beta=1/g_0^2$ (large glueball mass). In Eq. (1), besides the sum over repeated indices $\alpha,\beta=1,2,3,4$ (spin), $a=1,2,3$ (color) and $f=1,2,3\equiv u,d,s$ (flavor), the first sum runs over $u=(u^0,\vec u)=(u^0,u^1,u^2,u^3)\in\mathbb{Z}^4_o\equiv\{\pm1/2,\pm3/2,\pm5/2...\}\times\mathbb{Z}^3$, $\sigma=\pm1$ and $\mu=0,1,2,3$. Here, 0 labels the time component and direction 3 is sometimes called the z-direction. $e^{\mu=0,1,2,3}$ are lattice unit vectors. At a site $u\in\mathbb{Z}^4_o$, $\hat\psi_{a\alpha f}(u)$ are fermionic Grassmann fields (the hat means the presence or absence of a bar) and we refer to $\alpha=1,2$ as *upper* spin indices and $\alpha=3,4$ (equivalently, $+$ or $-$ respectively) as *lower* ones. For $\kappa=0$, $\langle\psi_{\ell_1}(x)\,\bar\psi_{\ell_2}(y)\rangle=\delta_{\alpha_1,\alpha_2}\delta_{a_1a_2}\delta_{f_1f_2}\delta(x-y)$, and the Grassmann integral of monomials is given by Wicks theorem. For each nearest neighbor oriented bond $< u,u\pm e^\mu >$ there is an $\mathrm{SU}(3)_c$ matrix $U(g_{u,u\pm e^\mu})$ parameterized by the gauge group element $g_{u,u\pm e^\mu}$ and satisfying $U(g_{u,u+e^\mu})^{-1}=U(g_{u+e^\mu,u})$. We sometimes drop the U from the notation. $\chi(U(g_p))$ is the plaquette term. $M\equiv M(m,\kappa)=m+2\kappa$. Given κ, $m>0$ is chosen such that $M_{\alpha\beta}=\delta_{\alpha\beta}$, i.e. $m=1-2\kappa\lesssim 1$. For more details about notation and conventions, see Ref. [9].

The physical quantum mechanical Hilbert space $\mathcal{H}$ and the E-M operators H and P^j, $j=1,2,3$, are defined via Feynman-Kac as in Refs. [6, 7]. Polymer expansion methods (see Refs. [6, 14, 15]) ensure the thermodynamic limit of correlations exists and truncated correlations have exponential tree decay. The limiting correlations are lattice translational invariant and extend to analytic functions in the global coupling parameters κ and $\beta=1/g_0^2$, and also in any finite number of local coupling parameters. For gauge-invariant F and G restricted to $u^0=1/2$, we have the F-K formula

$$(G,\check T_0^{x^0}\check T_1^{x^1}\check T_2^{x^2}\check T_3^{x^3}F)_{\mathcal{H}}=\langle[T_0^{x^0}\vec T^{\vec x}F]\Theta G\rangle\,, \tag{2}$$

where $T_0^{x^0}$, $T_i^{x^i}$, $i=1,2,3$, denote translation of the functions of Grassmann and gauge variables by $x^0=0,1,\ldots$, $\vec x=(x^1,x^2,x^3)\in\mathbb{Z}^3$, $T^{\vec x}=T_1^{x^1}T_2^{x^2}T_3^{x^3}$ and Θ is an antilinear, order reversing operator which involves time reflection (see Ref. [6]). In Eq. (2), we do not distinguish between Grassmann, gauge variables (rhs) and their associated Hilbert space vectors (lhs) in our notation. As linear operators in $\mathcal{H}$, $\check T_{\mu=0,1,2,3}$ are mutually commuting; $\check T_0$ is self-adjoint, with $-1\le\check T_0\le 1$, and $\check T_{j=1,2,3}$ are unitary. So, $\check T_j=e^{iP^j}$ defines the self-adjoint momentum operator $\vec P=(P^1,P^2,P^3)$ with spectral points $\vec p\in\mathbf{T}^3\equiv(-\pi,\pi]^3$ and $\check T_0^2=e^{-2H}\ge 0$ defines the self-adjoint energy operator $H\ge 0$. We call a point in the E-M spectrum with $\vec p=\vec 0$ a mass. Also, we let $\mathcal{E}(\lambda^0,\vec\lambda)$ be the product of the spectral families of $\check T_0$, P^1, P^2 and P^3.

We now briefly state our results. We show the existence of 36 eightfold way meson states (of asymptotic mass $-2\ln\kappa$) and determine their masses and dispersion curves exactly. Besides the usual quantum numbers associated with the $\mathrm{SU}(3)_f$ symmetry (total isospin, its third component, total hypercharge and quadratic Casimir), these states also carry spin labels. To define the components of the total spin operator in the Grassmann algebra, we use the symmetries of $\pi/2$ rotations about the spatial coordinate axes. Acting on improper zero momentum meson states these agree with the infinitesimal generators of the continuum. The total spin and its z component are the spin labels. The 36 eightfold way mesons are comprised of the spin 0 pseudo-scalar meson flavor singlet and octet and the spin 1 vector meson flavor singlet and octet. Charge conjugation $\mathcal{C}$ leaves invariant each of the singlets and octets so that these multiplets contain their anti-particles. The mesons have asymptotic mass of $\approx -2\ln\kappa$, and their existence is manifested by isolated dispersion curves in the E-M spectrum. Moreover, we show the meson spectrum is the only spectrum in $\mathcal{H}_e$ up to near the two-meson threshold of $\approx -4\ln\kappa$. Up to and including $\mathcal{O}(\kappa^4)$, within the pseudo-scalar meson flavor singlet and octet the masses are equal; the states in the vector meson flavor singlet and octet also have the same mass. However, by an explicit calculation, there is a state independent $\mathcal{O}(\kappa^4)$ mass splitting between the vector and pseudo-scalar states for each member of the octet and singlet. We use a correlation subtraction method to show that the only spectrum in $\mathcal{H}_e$ is generated by the eightfold way mesons up to near the two-meson threshold. Our result, combined with the one of Refs. [9, 10], shows that the only spectrum in all $\mathcal{H}$ is generated by the eightfold way mesons and baryons. Thus we have shown confinement up to near the two-meson threshold ($\approx -4\ln\kappa$).

It is worth noting that, as in Refs. [9, 10], our results are rigorously obtained using the hyperplane decoupling method and a spectral representation for the two-point function for the composite meson fields. For a pedagogical presentation of the basic principles of the hyperplane decoupling method see Ref. [15]. Besides, in our method, the form, the multiplicities, and also the gauge invariance (manifestation of confinement) of the mesonic excitations emerge naturally without any a priori guesswork. Also, we show that the masses are given by convergent expansions and, in fact, analytic in κ and β; thus we have controlled the expansion to all orders in κ and β (κ small, and $\beta << \kappa$). We emphasize we have determined the exact inverse propagator and have related the singularities of the Fourier transform of the propagator to points in the energy-momentum spectrum.

The paper is organized as follows. In section II, we use the decoupling of hyperplane method to reveal the basic excitation states. We also introduce a spectral representation for the the two-point correlation and explain our strategy to detect points in the energy-momentum spectrum. In section III, we pass, by an orthogonal transformation, from the basic excitation state basis, to a new basis, namely the particle basis, and identify the new states with the eightfold way mesons. In section IV, we obtain convergent expansions for the mesons masses, dispersion curves and their multiplicities. In section V, using a correlation subtraction method, we show that the only spectrum in all $\mathcal{H}_e$ is generated by the eightfold way mesons. Finally, in section VI we make some concluding remarks.

II. BASIC EXCITATION MESON STATES

We use the hyperplane decoupling method to obtain the basic excitation meson fields. Appropriate linear combinations of these fields are later identified with the eightfold way mesons. Also, we outline our strategy to find an appropriate two-point function and to detect particle masses and dispersion curves. We begin by defining the general truncated two-point function, for arbitrary $M, L \in \mathcal{H}_e$,

$$\mathcal{G}_{ML}(u,v) = \langle M(u)L(v)\rangle_T = \langle M(u)L(v)\rangle - \langle M(u)\rangle\langle L(v)\rangle . \tag{3}$$

We apply the decoupling of hyperplane method to the correlation of Eq. (3). This method consists of replacing κ by $\kappa_p \in \mathbb{C}$ in the action of Eq. (1) for each bond connecting adjacent temporal hyperplanes that separate u^0 and v^0. We expand $\mathcal{G}_{ML}(u,v)$ and we show that the coefficients of κ_p^0 and κ_p^1 are zero. As seen below, the second κ_p derivative at $\kappa_p = 0$ reveals the form of the basic excitation fields and the appropriate two-point function $\mathcal{G}$. Intuitively, we pick up a decay factor of κ_p for each vanishing κ_p derivative at $\kappa_p = 0$. Given the appropriate $\mathcal{G}$, this shows an analyticity domain in p^0 in the strip $|\mathrm{Im}\, p^0| < -(2-\epsilon)\ln\kappa$. We introduce Λ, the convolution inverse of $\mathcal{G}$ and show that it has a faster temporal decay than $\mathcal{G}$. Thus, its Fourier transform $\tilde{\Lambda}(p)$, $\tilde{\mathcal{G}}(p)\tilde{\Lambda}(p) = 1$, has a larger analyticity domain in p^0 than $\tilde{\mathcal{G}}(p)$, which turns out to be the strip $|\mathrm{Im}\, p^0| < -(4-\epsilon)\ln\kappa$. Then,

$$\tilde{\Lambda}^{-1}(p) = [\mathrm{cof}\,\tilde{\Lambda}(p)]^t / \det[\tilde{\Lambda}(p)]\,,$$

provides a meromorphic extension of $\tilde{\mathcal{G}}(p)$. The singularities of $\tilde{\Lambda}^{-1}(p)$ are solutions $\omega(\vec{p})$ of the equation

$$\det[\tilde{\Lambda}(p^0 = iw(\vec{p}), \vec{p})] = 0\,. \tag{4}$$

The solutions $w(\vec{p})$ will be shown to be the meson dispersion curves and the masses correspond to $w(\vec{p} = \vec{0})$.

To apply the hyperplane decoupling method, it is more convenient to use a duplicate of variable representation (see for more details Ref. [14]) for the truncated two-point function of Eq. (3). Letting $\hat{\psi}'$ and g' denote the duplicate field variables, we have, for $\mathcal{S} \equiv \mathcal{S}(\psi, \bar{\psi}, g)$ and $\mathcal{S}' \equiv \mathcal{S}(\psi', \bar{\psi}', g')$,

$$\begin{aligned} \mathcal{G}_{ML}(u,v) &= \frac{1}{2Z^2}\int [M(u) - M'(u)]\,[L(v) - L'(v)] \\ &\quad \times e^{-\mathcal{S}-\mathcal{S}'} d\psi d\bar{\psi} d\mu(g) d\psi' d\bar{\psi}' d\mu(g') \\ &\equiv \langle\langle [M(u) - M'(u)][L(v) - L'(v)]\rangle\rangle\,. \end{aligned} \tag{5}$$

Expanding the rhs of Eq. (5) in κ_p, we get $\mathcal{G}^{(0)}_{ML}(u,v) = 0$ (the coefficient of κ^r is denoted by the superscript $^{(r)}$), since the fields are on different sides of the hyperplane decouple. Next, $\mathcal{G}^{(1)}_{ML}(u,v) = 0$, noting that each expectation factorizes and each factor has an odd number of fermion fields. Finally, considering the second derivative of $\mathcal{G}$, using the gauge integral $\mathcal{I}_2 \equiv \int g_{a_1b_1} g^{-1}_{a_2b_2} d\mu(g) = (1/3)\delta_{a_1b_2}\delta_{a_2b_1}$ (Peter-Weyl) and, for the time ordering $u^0 \leq p < v^0$,

$$\begin{aligned}\mathcal{G}^{(2)}_{ML}(u,v) \;&=\; \textstyle\sum_{\vec{w}} \langle [M(u) - \langle M(u)\rangle]\, \bar{\mathcal{M}}_{\vec{\gamma}\vec{g}}(p,\vec{w})\rangle^{(0)} \\ &\quad \times \langle \mathcal{M}_{\vec{\gamma}\vec{g}}(p+1,\vec{w})\,[L(v) - \langle L(v)\rangle]\rangle^{(0)}\,,\end{aligned} \tag{6}$$

where

$$\begin{aligned}\bar{\mathcal{M}}_{\vec{\gamma}\vec{g}} &= \tfrac{1}{\sqrt{3}}\bar{\psi}_{a,\gamma_\ell,g_1}\psi_{a,\gamma_u,g_2}\,, \\ \mathcal{M}_{\vec{\gamma}\vec{g}} &= \tfrac{1}{\sqrt{3}}\psi_{a,\gamma_\ell,g_1}\bar{\psi}_{a,\gamma_u,g_2}\end{aligned} \tag{7}$$

and γ_u (γ_ℓ) refers to upper (lower) spin indices, $\vec{\gamma} = (\gamma_\ell,\gamma_u)$ and $\vec{g} = (g_1,g_2)$. Note that the fields in Eq. (7) are gauge invariant (colorless) local composite of fermion fields. It will be seen that the fields $\bar{\mathcal{M}}$ of Eq. (7) are basic excitation creating fields. Linear combinations of these fields correspond to meson particles. We refer to the basis generated by the $\bar{\mathcal{M}}$ as the individual spin and isospin basis (individual basis, for short) and denote by $\mathcal{H}_{\bar{\mathcal{M}}} \subset \mathcal{H}_e$ the subspace generated by the $\bar{\mathcal{M}}$'s. Charge conjugation $\mathcal{C}$, which is a symmetry of the model, transforms the $\bar{\mathcal{M}}$ as follows; $\mathcal{C}\bar{\mathcal{M}}_{31,f_1f_2} = \bar{\mathcal{M}}_{42,f_2f_1}$, $\mathcal{C}\bar{\mathcal{M}}_{42,f_1f_2} = \bar{\mathcal{M}}_{31,f_2f_1}$, $\mathcal{C}\bar{\mathcal{M}}_{41,f_1f_2} = \bar{\mathcal{M}}_{41,f_2f_1}$, $\mathcal{C}\bar{\mathcal{M}}_{32,f_1f_2} = \bar{\mathcal{M}}_{32,f_2f_1}$. From this, we see that $\mathcal{H}_{\bar{\mathcal{M}}}$ is invariant under $\mathcal{C}$.

From the vanishing of the zeroth and first hyperplane derivatives, using joint analyticity in all κ_p's and Cauchy estimates, we obtain the global bound

$$|\mathcal{G}_{ML}(u,v)| \;\leq\; \text{const}\,|\kappa|^{2|u^0-v^0|+2|\vec{u}-\vec{v}|}\,, \tag{8}$$

where $|\vec{u}-\vec{v}| = \sum_{i=1}^{3}|u^i - v^i|$ and recall $\mathcal{G}$ is jointly analytic in κ and β. The spatial decay is obtained by applying the hyperplane method to spatial hyperplanes. From Eq. (8), the analyticity of $\tilde{\mathcal{G}}_{ML}(p)$ in the p^0-strip $|\mathrm{Im}\,p^0| < -(2-\epsilon)\ln\kappa$ follows.

For closure, meaning that correlations on the lhs and rhs of Eq. (6) are the same, we take the fields $M = \mathcal{M}_{\vec{\alpha}\vec{f}}$ and $L = \bar{\mathcal{M}}_{\vec{\beta}\vec{h}}$ in Eq. (6) to obtain, for $u^0 \leq p < v^0$, using that $\langle\bar{\mathcal{M}}(u)\rangle = \langle\mathcal{M}(u)\rangle = 0$ (by parity symmetry $\mathcal{P}$ of Ref. [7]),

$$\begin{aligned}\langle\mathcal{M}_{\vec{\alpha}\vec{f}}(u)\bar{\mathcal{M}}_{\vec{\beta}\vec{h}}(v)\rangle^{(2)} \;&=\; \textstyle\sum_{\vec{w}}\langle\mathcal{M}_{\vec{\alpha}\vec{f}}(u)\bar{\mathcal{M}}_{\vec{\gamma}\vec{g}}(p,\vec{w})\rangle^{(0)} \\ &\quad\times\langle\mathcal{M}_{\vec{\gamma}\vec{g}}(p+1,\vec{w})\bar{\mathcal{M}}_{\vec{\beta}\vec{h}}(v)\rangle^{(0)}\,,\end{aligned} \tag{9}$$

which we call the product structure property and write schematically as $\mathcal{G}^{(2)}_{ML}(u,v) = [\mathcal{G}^{(0)}_{M\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\mathcal{M}L}](u,v)$. A similar expression is obtained for the other time ordering, i.e. $u^0 > p \geq v^0$. The importance of the product structure is that it feeds into the formula for the second derivative of Λ showing that it is zero, giving rise to the faster decay of Λ.

The appropriate two-point function for the excitation fields is given by $[\mathcal{G}_{\mathcal{M}_\ell\bar{\mathcal{M}}_{\ell'}}(u,v) \equiv \mathcal{G}_{\ell\ell'}(u,v) = \mathcal{G}_{\ell\ell'}(x = u-v)]$,

$$\begin{aligned}\mathcal{G}_{\ell\ell'}(x) \;&=\; \left\langle\mathcal{M}_\ell(u)\bar{\mathcal{M}}_{\ell'}(v)\right\rangle\chi_{u^0\leq v^0} \\ &\quad+\left\langle\bar{\mathcal{M}}_\ell(u)\mathcal{M}_{\ell'}(v)\right\rangle^*\chi_{u^0>v^0}\,,\end{aligned} \tag{10}$$

and $\ell' = (\vec{\beta},\vec{h}\,)$ are collective indices, and χ is the characteristic function. Time-reversal $\mathcal{T}$ and parity $\mathcal{P}$ of Ref. [7], for fixed u and v, shows that $\mathcal{G}(u,v)$ is self-adjoint.

Note also that, for fixed u and v, the dimension of the matrix $\mathcal{G}$ is $36 = (2\times3)^2$ where 2 is the spin space dimension and 3 is the number of flavors.

By the F-K formula with $u^0 \neq v^0$ and using the spectral representation of the E-M operators, $\mathcal{G}(x)$ can be written as ($x^0 \neq 0$), $\bar{\mathcal{M}}_\ell \equiv \bar{\mathcal{M}}_\ell(1/2,\vec{0})$,

$$\mathcal{G}_{\ell\ell'}(x) = \int_{-1}^{1}\int_{T^d}\lambda_0^{|x^0|-1}e^{i\vec{\lambda}.\vec{x}}d(\bar{\mathcal{M}}_\ell,\mathcal{E}(\lambda_0,\vec{\lambda})\bar{\mathcal{M}}_{\ell'})_{\mathcal{H}}\,, \tag{11}$$

and is an even function of $\vec{x}$ by parity symmetry. From Eq. (11), we see that the $\bar{\mathcal{M}}$ creates particles. After separating the equal time contribution, the Fourier transform $\tilde{\mathcal{G}}_{\ell\ell'}(p) = \sum_{x\in\mathbb{Z}^4}\mathcal{G}_{\ell\ell'}(x)e^{-ip.x}$, satisfies

$$\tilde{\mathcal{G}}_{\ell\ell'}(p) = \tilde{\mathcal{G}}_{\ell\ell'}(\vec{p}) + (2\pi)^3\int_{-1}^{1}f(p^0,\lambda^0)d_{\lambda^0}d\alpha_{\vec{p},\ell\ell'}(\lambda^0)\,, \tag{12}$$

where

$$d\alpha_{\vec{p},\ell\ell'}(\lambda^0) = \int_{\mathbb{T}^3}\delta(\vec{p}-\vec{\lambda})d_{\lambda^0}d_{\vec{\lambda}}(\bar{\mathcal{M}}_\ell,\mathcal{E}(\lambda^0,\vec{\lambda})\bar{\mathcal{M}}_{\ell'})_{\mathcal{H}},$$

with $f(x,y) = (e^{ix}-y)^{-1} + (e^{-ix}-y)^{-1}$, and we have set $\tilde{\mathcal{G}}(\vec{p}) = \sum_{\vec{x}}e^{-i\vec{p}.\vec{x}}\mathcal{G}(x^0=0,\vec{x})$.

From Eq. (12), we see that points in the E-M spectrum are detected as singularities on the imaginary p^0 axis of $\tilde{\mathcal{G}}_{\ell\ell'}(p)$. These singularities are given by the solutions of Eq. (4). The convolution inverse Λ of $\mathcal{G}$ is defined by the Neumann series $\Lambda = (1+\mathcal{G}_d^{-1}\mathcal{G}_n)^{-1}\mathcal{G}_d^{-1} = \sum_i(-1)^i\left[\mathcal{G}_d^{-1}\mathcal{G}_n\right]^i\mathcal{G}_d^{-1}$ where $\mathcal{G}_d$ is given by $\mathcal{G}_{d,\ell\ell'}(u,v) = \mathcal{G}_{d,\ell\ell'}(u,u)\delta_{\ell\ell'}\delta_{u,v}$, with $\mathcal{G} = \mathcal{G}_d + \mathcal{G}_n$. Λ is well defined using the global bound of Eq. (8) and noting that $\mathcal{G}_d$ is 1 at $\kappa = 0$ by the normalization of the $\bar{\mathcal{M}}$ fields of Eq. (7).

Taking the hyperplane derivative, using the relation $\mathcal{G}\Lambda = 1 = \Lambda\mathcal{G}$, the Leibniz formula $\partial^r\Lambda = -\sum_{s=0}^{r-1}\binom{r}{s}\Lambda\partial^{r-s}\mathcal{G}\partial^s\Lambda$, we obtain $\Lambda^{(r=0,1)}(u,v) = 0$, for $|u^0-v^0| \geq 1$. Furthermore, $\Lambda^{(2)}(u,v) = -[\Lambda^{(0)}\mathcal{G}^{(2)}\Lambda^{(0)}](u,v)$ and, by the product structure of Eq. (9), is zero for $|u^0-v^0| > 1$. Also, by imbalance of fermion fields $\mathcal{G}^{(3)}(u,v) = 0$ leading to $\Lambda^{(3)}(u,v) = 0$,

$|u^0 - v^0| > 1$. From this follows the global bound

$$|\Lambda(u,v)| \le c|\kappa|^2 |\kappa|^{4(|u^0-v^0|-1)+2|\vec{u}-\vec{v}|}\ ,\ |u^0 - v^0| \ge 1\ , \tag{13}$$

with the rhs replaced by const$\kappa^{2|\vec{u}-\vec{v}|}$, if $u^0 = v^0$, and the analyticity of $\tilde{\Lambda}(p)$ in the larger p^0-strip $|\mathrm{Im}p^0| < -(4-\epsilon)\ln\kappa$. Also, Λ is jointly analytic in κ and β.

III. PARTICLE BASIS: PSEUDOSCALAR AND VECTOR MESON

We now consider operators associated with the $\mathrm{SU}(3)_f$ symmetry and spin operators. For F a function of the Grassmann fields, we define I_j, the $j-th$ ($j = 1,2,3$) component of total isospin, by $A_j = \lim_{\theta\searrow 0}(\mathcal{W}(U)F - F)/(i\theta)$, with $\mathcal{W}(U)F = F(\{U\bar{\psi}\}, \{\bar{U}\psi\})$, where $U_j \equiv U_j(\theta) = \exp(i\lambda_j\theta/2)$, $j = 1,\ldots,8$ is an element of $\mathrm{SU}(3)_f$ and the λ_j are the usual Gell'Mann matrices of Refs. [1, 2]. For example, if $F = \bar{\psi}\psi$ and letting $i_j = \lambda_j/2$ then $I_jF = (i_j\bar{\psi})\psi - \bar{\psi}(\bar{i}_j\psi)$. The total hypercharge Y is defined as $2A_8/\sqrt{3}$ and the quadratic Casimir by $C_2 = \sum_{j=1}^{8} A_j^2$. The linear operator $\mathcal{W}(U)$ lifts to a unitary operator $\check{\mathcal{W}}(U)$ on $\mathcal{H}$ by using the F-K formula and the $\mathrm{SU}(3)_f$ symmetry. The generators $\check{A}_j = \lim_{\theta\searrow 0}(\check{\mathcal{W}}(U)F - F)/(i\theta)$ of the eight one-parameter subgroups are self-adjoint operators in $\mathcal{H}$. $\check{I}_3$, $\check{I}^2 = \check{I}_1^2 + \check{I}_2^2 + \check{I}_3^2$, $\check{Y}$, $\check{C}_2$ are mutually commuting and their eigenvalues are quantum numbers which are used to label the states. The total spin operators J_x, J_y and J_z are also defined similarly on the Grassmann algebra only, with $U = U_2 \oplus U_2$, $U_2 = \exp(i\theta\sigma^j/2) \in SU(2)$. On improper zero-momentum states of Eq. (14) this infinitesimal generator definition agrees with that obtained by taking $(2/i\pi)\ln W(U)$, with $\theta = \pi/2$. For the rotation angle $\theta = \pi/2$ we have a lattice symmetry and the transformation of the spin components of the $\hat{\psi}$'s agrees with the continuum. The eigenvalues of J_z and J of the eigenvalue $J(J+1)$ of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ are also used to label the states.

We recall that the mesons dispersion relations $w(\vec{p})$ are given by the solutions of Eq. (4) (for $\vec{p} = \vec{0}$ we have the masses). We remark that, due to the determinant in Eq. (4) we are free to take any new basis related to the individual spin or isospin basis by a real orthogonal transformation. By fully exploiting the $\mathrm{SU}(3)_f$ symmetry and additional symmetries (more details ahead) the pseudo-scalar and vector mesons emerge from our analysis as linear combinations of the basic excitation fields in Eq. (7). The linear combinations are associated with a real orthogonal transformation that diagonalizes $\tilde{\mathcal{G}}(p^0, \vec{p} = \vec{0})$. In the sequel, we introduce a new isospin and spin basis and show the conventional connection with the pseudo-scalar and vector mesons. For this purpose, for fixed $\vec{\alpha} = (\alpha_\ell, \alpha_u)$ we decompose the individual isospin basis into the direct sum of irreducible representations of $\mathrm{SU}(3)_f$, precisely, a 1-dimensional flavor singlet (denoted by $\bar{\mathcal{M}}_{\vec{\alpha}}^0$, with $C_2 = 0$) and an 8-dimensional octet ($\{\bar{\mathcal{M}}_{\vec{\alpha}}^k\}_{k=1}^8$, with $C_2 = 3$) and the labelling distinguishes between them. The new basis elements are listed below:

$$\begin{aligned}
\bar{\mathcal{M}}_{\vec{\alpha}}^0 &= \tfrac{1}{3}(\bar{\psi}_{a,\alpha_\ell,u}\psi_{a,\alpha_u,u} + \bar{\psi}_{a,\alpha_\ell,d}\psi_{a,\alpha_u,d} \\
&\quad + \bar{\psi}_{a,\alpha_\ell,s}\psi_{a,\alpha_u,s}) \\
\bar{\mathcal{M}}_{\vec{\alpha}}^1 &= \tfrac{1}{3\sqrt{2}}(\bar{\psi}_{a,\alpha_\ell,u}\psi_{a,\alpha_u,u} + \bar{\psi}_{a,\alpha_\ell,d}\psi_{a,\alpha_u,d} \\
&\quad - 2\bar{\psi}_{a,\alpha_\ell,s}\psi_{a,\alpha_u,s}) \\
\bar{\mathcal{M}}_{\vec{\alpha}}^2 &= \tfrac{1}{\sqrt{6}}(\bar{\psi}_{a,\alpha_\ell,u}\psi_{a,\alpha_u,u} - \bar{\psi}_{a,\alpha_\ell,d}\psi_{a,\alpha_u,d}) \\
\bar{\mathcal{M}}_{\vec{\alpha}}^3 &= \tfrac{1}{\sqrt{3}}\bar{\psi}_{a,\alpha_\ell,u}\psi_{a,\alpha_u,d} \\
\bar{\mathcal{M}}_{\vec{\alpha}}^4 &= \tfrac{1}{\sqrt{3}}\bar{\psi}_{a,\alpha_\ell,u}\psi_{a,\alpha_u,s} \\
\bar{\mathcal{M}}_{\vec{\alpha}}^5 &= \tfrac{1}{\sqrt{3}}\bar{\psi}_{a,\alpha_\ell,d}\psi_{a,\alpha_u,s} \\
\bar{\mathcal{M}}_{\vec{\alpha}}^6 &= \tfrac{1}{\sqrt{3}}\bar{\psi}_{a,\alpha_\ell,d}\psi_{a,\alpha_u,u}\ , \\
\bar{\mathcal{M}}_{\vec{\alpha}}^7 &= \tfrac{1}{\sqrt{3}}\bar{\psi}_{a,\alpha_\ell,s}\psi_{a,\alpha_u,u}\ , \\
\bar{\mathcal{M}}_{\vec{\alpha}}^8 &= \tfrac{1}{\sqrt{3}}\bar{\psi}_{a,\alpha_\ell,s}\psi_{a,\alpha_u,d}\ .
\end{aligned} \tag{14}$$

Using isospin orthogonality relations, the vectors in Eq. (14) are eigenvectors of the generators $\check{I}_3$, $\check{I}_2$, $\check{Y}$, $\check{C}_2$ and the superscript $k = (I_3, I, Y, C_2)$ is a collective index for the eigenvalues of those generators. More explicitly, we have the identifications: $(0,0,0,0)$, $(0,0,0,3)$, $(0,1,0,3)$, $(1,1,0,3)$, $(1/2,1/2,1,3)$, $(-1/2,1/2,1,3)$, $(-1,1,0,3)$, $(-1/2,1/2,-1,3)$ and $(1/2,1/2,-1,3)$, for $k = 0,1,\ldots,8$, respectively. We refer to this basis as the total isospin, individual spin basis. With fixed $\vec{\alpha}$, using the $\bar{\mathcal{M}}_{\vec{\alpha}\vec{f}}$ ordering $\vec{f} = (u,u)$, (d,d), (s,s), (u,d), (u,s), (d,s), (d,u), (s,u), (s,d), the transformation matrix from the individual basis to this new basis is explicitly given by $B = B_3 \oplus I_6$ with $(B_3)_{11} = (B_3)_{12} = (B_3)_{13} = 1/\sqrt{3}$, $(B_3)_{21} = (B_3)_{22} = -\sqrt{2}(B_3)_{23} = 1/\sqrt{6}$, $(B_3)_{31} = -(B_3)_{32} = 1/\sqrt{2}$ and $(B_3)_{33} = 0$. The two-point function in the total isospin, individual spin basis decomposes into the direct sum of 8 identical 4×4 blocks and one 4×4 block associated with the flavor octet and singlet, respectively. The product structure still holds in the new basis, as well as the global bound of Eq. (13).

The 4×4 block can be further reduced by using $\mathcal{G}_p$, a generalized local $\mathcal{G}$-parity symmetry, which is a composition of charge conjugation $\mathcal{C}$ and discrete $\mathrm{SU}(3)_f$ symmetry, namely permutation of flavor indices. More explicitly, for $k = 0,1,2$, $\mathcal{G}_p = \mathcal{C}$, since the isospin index for the anti-quark and quark pair in $\bar{\mathcal{M}}^k$ is the same. For the other vectors, $\bar{\mathcal{M}}^k$ ($k = 3,\ldots,8$), in Eq. (14) we need to compose $\mathcal{C}$ with a permutation matrix $P \in SU(3)_f$. For example, for $\bar{\mathcal{M}}^3$, the non-vanishing elements of P are $(P)_{12} = (P)_{21} = -(P)_{33} = 1$. We decompose the space $\bar{\mathcal{M}}_{\vec{\alpha}}^k$ into eigenvectors of $\mathcal{G}_p$ given by (suppressing all but the spin index) $(\bar{\mathcal{M}}_{31} + \bar{\mathcal{M}}_{42})/\sqrt{2}$ with eigenvalue 1 and $\{\bar{\mathcal{M}}_{32}\ , (\bar{\mathcal{M}}_{31} - \bar{\mathcal{M}}_{42})/\sqrt{2}\ , \bar{\mathcal{M}}_{41}\}$ with eigenvalue -1. With this decomposition, the 36×36 two-point function matrix, for fixed u and v, reduces to a direct sum of nine

4×4 blocks (8 identical blocks for the octet states) each one decomposing as $(1\times 1)\oplus(3\times 3)$. We denote the eigenvectors of $\mathcal{G}_p$ by $\bar{\mathcal{M}}_{\mathcal{J}}$ which are related to $\bar{\mathcal{M}}_{\vec{\alpha}}$ by a real 4×4 orthogonal transformation (A_4) with nonvanishing elements $(A_4)_{11}=(A_4)_{12}=(A_4)_{31}=-(A_4)_{32}=1/\sqrt{2}$, $(A_4)_{24}=(A_4)_{43}=1$. Here we are adopting the following ordering of $\vec{\gamma}=(\gamma_\ell,\gamma_u)$: $(3,1),(4,2),(4,1),(3,2)$.

It turns out that the decomposition provided by $\mathcal{G}_p$ is precisely the decomposition into the total spin pseudo-scalar and vector excitations. The vectors $\bar{\mathcal{M}}_{\mathcal{J}}$ are eigenvectors of $\vec{J}^2$ and J_z, with eigenvalues $(0,0)$, $(1,1)$, $(1,0)$, $(1,-1)$, respectively. $\mathcal{G}_p$ does not distinguish between the states $(J,J_z)=(1,1),(1,0),(1,-1)$, but spin does. We will label the vectors $\bar{\mathcal{M}}_{\mathcal{J}}$ of the total isospin, total spin basis by $\mathcal{J}=(J,J_z)$ in the order given above and denote the two-point correlation in this basis by $G_{\mathcal{J}\mathcal{J}'}(x)$ and its convolution inverse by $\Gamma_{\mathcal{J}\mathcal{J}'}(x)$. We note that the same global bounds of Eq. (13) hold for $\Gamma_{\mathcal{J}}(x)$ as for $\Lambda(x)$ as again only orthogonal basis transformations are involved.

We obtain further relations between the elements of $G_{\mathcal{J}\mathcal{J}'}(x)$ by exploiting a new local anti-linear symmetry which we call spin flip, denoted by $\mathcal{F}_s$ (see Ref. [9]). This symmetry is the composition $-i\mathcal{T}\mathcal{C}\mathrm{T}$ of non-local, linear time reflection T, local linear charge conjugation $\mathcal{C}$ and nonlocal anti-linear time reversal $\mathcal{T}$ symmetries. Using the spin flip symmetry, we obtain the structure $(G_{\mathcal{J}\mathcal{J}'})=A_1\oplus A_3$ with A_1 real, associated with $J=0$, and A_3 a 3×3 self-adjoint matrix associated with $J=1$ and obeying $(A_3)_{11}=(A_3)_{33}$ and $(A_3)_{12}=(A_3)_{23}$. This matrix structure carries over to $\Gamma_{\mathcal{J}\mathcal{J}'}$, $\tilde{G}_{\mathcal{J}\mathcal{J}'}(p^0=i\chi,\vec{p})$ and $\tilde{\Gamma}_{\mathcal{J}\mathcal{J}'}(p^0=i\chi,\vec{p})$ (χ real). Summarizing, the matrix-valued two-point correlation and its convolution inverse reduce to a direct sum of nine 4×4 blocks (eight identical ones associated to the octet states), each one decomposing as $(1\times 1)\oplus(3\times 3)$, corresponding to total spin $J=0$ and $J=1$, respectively.

We make some remarks concerning the conventional connection with particles, i.e. the pseudo-scalar and vector mesons. For the pseudo-scalar mesons, $\bar{\mathcal{M}}_{(0,0)}$, we have the identifications $\bar{\mathcal{M}}^0_{(0,0)}=\eta'$, $\bar{\mathcal{M}}^1_{(0,0)}=\eta$, $\bar{\mathcal{M}}^2_{(0,0)}=\pi^0$, $\bar{\mathcal{M}}^3_{(0,0)}=\pi^+$, $\bar{\mathcal{M}}^4_{(0,0)}=K^+$, $\bar{\mathcal{M}}^5_{(0,0)}=K^0$, $\bar{\mathcal{M}}^6_{(0,0)}=\pi^-$, $\bar{\mathcal{M}}^7_{(0,0)}=K^-$, $\bar{\mathcal{M}}^8_{(0,0)}=\bar{K}^0$, recalling the fields of Eq. (14). In a similar way, we can identify the vector mesons, except for ϕ and ω which seem to be best described as strong mixtures of $\bar{\mathcal{M}}^0_{(1,J_z)}$ and $\bar{\mathcal{M}}^1_{(1,J_z)}$. The procedure just described permits us to label each eightfold way meson state by the quantum numbers (I,I_3,Y) which coincide e.g. with those depicted in Fig. 5.11 of Ref. [2]. Note that η', η and π^0 are invariant under charge conjugation. Also, $\mathcal{C}\pi^\pm=\pi^\mp$, $\mathcal{C}K^\pm=K^\mp$ and $\mathcal{C}K^0=\bar{K}^0$. Hence, charge conjugation $\mathcal{C}$ changes the sign of the hypercharge Y and the third component of isospin I_3 of a pseudo-scalar state.

IV. EIGHTFOLD WAY MESON MASSES AND DISPERSION CURVES

We now turn to the exact determination of the eightfold way meson dispersion curves and masses. We determine explicitly the masses (dispersion curves) up to and including $\mathcal{O}(\kappa^4)$ ($\mathcal{O}(\kappa^2)$) and $\beta=0$; for this we need the low order in κ, short distance behavior of $\mathcal{G}$ and Λ, which is shown to be independent of isospin. Hence, fixing a member of the octet or singlet the mass-determining equation is $\det\tilde{\Gamma}_{\mathcal{J}\mathcal{J}'}(p^0=iM,\vec{0})=0$. $\tilde{\Gamma}_{\mathcal{J}\mathcal{J}'}$ is seen to be diagonal by using the symmetry of $\pi/2$ rotations about the e^3 axis. For each factor we have the single equation $\tilde{\Gamma}_{\mathcal{J}\mathcal{J}}=0$ and for notational simplicity we write $\tilde{\Gamma}_{\mathcal{J}}$. The relation $\tilde{\Gamma}_{(1,1)}=\tilde{\Gamma}_{(1,-1)}$ can also be obtained using e^1 reflections instead of $\mathcal{F}_s$.

The solution of $\det\tilde{\Gamma}_{\mathcal{J}}=0$ for all $\vec{p}$ runs out to infinity as κ goes to zero. To find the solutions of $\det\tilde{\Gamma}_{\mathcal{J}}=0$ without approximation, we employ the auxiliary function method (see Ref. [15]). We make a nonlinear transformation from p^0 to an auxiliary variable w and introduce an auxiliary matrix function $H_{\mathcal{J}\mathcal{J}'}(w,\kappa,\vec{p})$ (for the masses we take $\vec{p}=\vec{0}$) to bring the solution for the nonsingular part $w(\vec{p})+2\ln\kappa$ of the dispersion curves from infinity to close to $w=0$ for small κ. With this function we can cast the problem of determining dispersion curves and masses into the framework of the analytic implicit function theorem. To this end we introduce the new variable, with $c_2(\vec{p})\equiv c_2\sum_{j=1}^3 2\cos p^j$ and $c_2=1/4$,

$$w=1-c_2(\vec{p})\kappa^2-\kappa^2e^{-ip^0} \tag{15}$$

and the auxiliary function $H_{\mathcal{J}}(w,\kappa,\vec{p})$ such that $\tilde{\Gamma}_{\mathcal{J}}(p^0,\vec{p}) = H_{\mathcal{J}}\left(w=1-c_2(\vec{p})\kappa^2-\kappa^2e^{-ip^0},\kappa,\vec{p}\right)$. $H_{\mathcal{J}}(w,\kappa,\vec{p})$ is defined by, using $\Gamma(x^0,\vec{x})=\Gamma(-x^0,\vec{x})$ (by time reversal and parity)

$$\begin{aligned}H_{\mathcal{J}\mathcal{J}'}(w,\kappa,\vec{p}) \;&=\; \sum_{\vec{x}}\Gamma_{\mathcal{J}\mathcal{J}'}(0,\vec{x})e^{-i\vec{p}.\vec{x}}+\sum_{n\ge 1,\vec{x}}\Gamma_{\mathcal{J}\mathcal{J}'}(n,\vec{x})\\ &\times e^{-i\vec{p}.\vec{x}}\left[\left(\tfrac{1-w-c_2(\vec{p})\kappa^2}{\kappa^2}\right)^n+\left(\tfrac{\kappa^2}{1-w-c_2(\vec{p})\kappa^2}\right)^n\right],\end{aligned}$$

with $\mathcal{J}=\mathcal{J}'$. By the global bounds of Λ of Eq. (13), H is jointly analytic in w and κ for $|w|$, $|\kappa|$ small.

The mass determining equation becomes $H_{\mathcal{J}}(w,\kappa)\equiv H_{\mathcal{J}\mathcal{J}}(w,\kappa,\vec{p}=\vec{0})=0$. In the sequel, we explicitly determine the masses up to and including $\mathcal{O}(\kappa^4)$. For this, we separate all terms in H up to $\mathcal{O}(\kappa^4)$ and we need $\Gamma(x^0=n,\vec{x})/\kappa^{2n}$ up to and including order κ^4. The normalization condition $G^{(0)}(x=0)=1$ implies $\Gamma^{(0)}(x=0)=1$ and by a simple argument the product formula gives $G(x=e^0)=\kappa^2+\mathcal{O}(\kappa^6)$, which implies $\Gamma(x=e^0)=-\kappa^2+\mathcal{O}(\kappa^6)$. Backtracking paths, such as $0\to e^0\to 0\to e^0$, do not contribute to $G(e^0)$ using the property ($\mu=0,1,2,3$, $\epsilon=\pm 1$): $\Gamma^{\epsilon e^\mu}\Gamma^{-\epsilon e^\mu}=0$, which we refer as the *come and go* property. Other contributions are found by explicit calculation of coefficients of the hopping parameter expansion of $G(x)$ and arise from non-intersecting paths connecting the point 0 to x and using the Neumann series for $\Gamma(x)$. In non-intersecting paths each link of the path is composed of no more than two oppositely oriented bonds, since, taking into account $2n-$bonds ($n>1$), two by two oppositely oriented, we would have contractions of the fields in the expectations at 0 and x giving zero using the *come and go* property. For this purpose, we have derived a general formula for calculating non-intersecting path contributions given by (in the individual basis)

$$\langle \mathcal{M}_{\vec{\alpha}\vec{f}}(0)\bar{\mathcal{M}}_{\vec{\beta}\vec{f}'}(x)\rangle =_p \left(\frac{\kappa}{2}\right)^{2L} \delta_{\vec{f}\vec{f}'}\Gamma^{p}_{\alpha_\ell\beta_\ell}\Gamma^{-p}_{\beta_u\alpha_u} \qquad (16)$$

where $\vec{\alpha}=(\alpha_\ell,\alpha_u)$, $\vec{\beta}=(\beta_\ell,\beta_u)$, $\vec{f}=(f_1,f_2)$, $\vec{f}'=(f_3,f_4)$ and L is the length of the path. In non-intersecting paths only $\mathcal{I}_2$ occurs. The subscript p in Eq. (16) above means that we take only the contribution coming from a non-intersecting path, with consecutive points of the path linked by two overlapping bonds of opposite orientation. The notation $\Gamma^{p}_{\alpha\beta}$ ($\Gamma^{-p}_{\alpha\beta}$) means the $\alpha\,\beta$ element of the ordered product of Γ matrices along the path that connects 0 to x (x to 0). For example, if $x=e^0+e^1+e^2$ and the path is chosen such that $0\to e^0\to e^0{+}e^1\to e^0{+}e^1{+}e^2$, hence $\Gamma^p=\Gamma^0\Gamma^1\Gamma^2$ where we have used the notation $\Gamma^{\epsilon e^\rho}\equiv\Gamma^{\epsilon\rho}$. For the reversed path we take the product of the Γ matrices in the opposite direction, i.e. $\Gamma^{-p}=\Gamma^{-2}\Gamma^{-1}\Gamma^{-0}$. In general, if the path is as follows $0\to x_1\to x_2\to\ldots\to x_n\to x$ then $\Gamma^p\equiv\Gamma^{0\to x}=\Gamma^{x_1}\Gamma^{x_2-x_1}\ldots\Gamma^{x_n-x_{n-1}}\Gamma^{x-x_n}$ and $L=n+1$; $\Gamma^{-p}\equiv\Gamma^{x\to 0}=\Gamma^{-(x-x_n)}\Gamma^{-(x_n-x_{n-1})}\ldots\Gamma^{-(x_2-x_1)}\Gamma^{-x_1}$. Using Eq. (16), for ϵ, $\epsilon'=\pm1$, $j=1,2,3$ and $\beta=0$, we get the short distance behaviors of ($i=1,2$ below)

$$G_{\mathcal{J}}(x)=\begin{cases} 1+\mathcal{O}(\kappa^8) & ,\ x=0;\\ \kappa^2+\mathcal{O}(\kappa^6) & ,\ x=\epsilon e^0;\\ c_2\kappa^2+\mathcal{O}(\kappa^6) & ,\ x=\epsilon e^j;\\ \kappa^4+\mathcal{O}(\kappa^8) & ,\ x=2\epsilon e^0;\\ c_2\kappa^4+\mathcal{O}(\kappa^8) & ,\ x=2\epsilon e^j;\\ 2c_2\kappa^4+\mathcal{O}(\kappa^8) & ,\ x=\epsilon e^0+\epsilon' e^j;\\ c_2\delta_{J_z,0}\kappa^4+\mathcal{O}(\kappa^8) & ,\ x=\epsilon e^1+\epsilon' e^2;\\ \left[2c_2^2+\delta_{J_z,0}(\delta_{J,0}-\delta_{J,1})\right] & \\ \times\kappa^4+\mathcal{O}(\kappa^8) & ,\ x=\epsilon e^i+\epsilon' e^3\,; \end{cases}$$

and from the Neumann series

$$\Gamma_{\mathcal{J}}(x)=\begin{cases} 1+(2+6c_2^2)\kappa^4+\mathcal{O}(\kappa^8) & ,\ x=0;\\ -\kappa^2-\kappa^6+\mathcal{O}(\kappa^8) & ,\ x=\epsilon e^0;\\ -c_2\kappa^2+\mathcal{O}(\kappa^6) & ,\ x=\epsilon e^j\,;\\ (-c_2+c_2^2)\kappa^4+\mathcal{O}(\kappa^8) & ,\ x=2\epsilon e^j;\\ (-c_2\delta_{J_z,0}+2c_2^2) & \\ \times\kappa^4+\mathcal{O}(\kappa^8) & ,\ x=\epsilon e^1+\epsilon' e^2;\\ \delta_{J_z,0}(\delta_{J,0}-\delta_{J,1}) & \\ \times\kappa^4+\mathcal{O}(\kappa^8) & ,\ x=\epsilon e^i+\epsilon' e^3. \end{cases}$$

The other points listed in G but not appearing in Γ contribute to $\mathcal{O}(\kappa^6)$ in H. The improved decay for these points is due to explicit cancellations in the Neumann series and improve the global bounds obtained by the decoupling of hyperplane method.

Taking into account contributions of $H_{\mathcal{J}}$ up to and including $\mathcal{O}(\kappa^4)$ we can write $H_{\mathcal{J}}(w,\kappa)$ in the form, with $b_{\mathcal{J}}=-1+c_0+c_2(\vec{0})c_4$, $a_{\mathcal{J}}=(-3\delta_{J,1}-\delta_{J,0}+4!c_2^2)$: $c_0=2+6c_2^2$, $c_4=c_2-1$, $H_{\mathcal{J}}=w+b_{\mathcal{J}}\kappa^4-\kappa^4/(1-w)+a_{\mathcal{J}}\kappa^4+\kappa^6 r_{\mathcal{J}}(w,\kappa)$ where $a_{\mathcal{J}}$ is called an angle contribution coming from paths, in the κ expansion of G, of the form $0\to\epsilon e^i\to\epsilon e^i+\epsilon' e^j$, $ij=12,13,23$, $\epsilon,\epsilon'=\pm1$ and $r_{\mathcal{J}}(w,\kappa)$ is jointly analytic in w and κ. We see that $H_{\mathcal{J}}(0,0)=0$ and $(\partial H_{\mathcal{J}}/\partial w)(0,0)=1$ so that the analytic implicit function theorem applies and yields the analytic function $w_{\mathcal{J}}(\kappa)$ such that $H_{\mathcal{J}}(w_{\mathcal{J}}(\kappa),\kappa)=0$. The solution $w_{\mathcal{J}}(\kappa)$ has the form $w_{\mathcal{J}}(\kappa)=\kappa^4-(a_{\mathcal{J}}+b_{\mathcal{J}})\kappa^4+\mathcal{O}(\kappa^6)$. Returning to Eq. (15) the mass is given by

$$\begin{aligned} M_{\mathcal{J}}=\ln e^{-i(p^0=iM_{\mathcal{J}})} &= -2\ln\kappa+\ln(1-w_{\mathcal{J}}-c_2(\vec{0})\kappa^2)\\ &= -2\ln\kappa+d_2\kappa^2+d_4\kappa^4+\mathcal{O}(\kappa^6)\,, \end{aligned}$$

with $d_2=-6c_2=-3/2$ and $d_4=(-1+c_0+6c_2c_4)-\frac{1}{2}6^2c_2^2+a_{\mathcal{J}}=-2+a_{\mathcal{J}}$. Thus we see that the angle contribution $a_{\mathcal{J}}$ gives rise to a mass splitting between the total spin one and total spin zero states and is given by $M_{(1,J_z)}-M_{(0,0)}=2\kappa^4+\mathcal{O}(\kappa^6)$.

For the determination of the dispersion curves, we recall the block decomposition of $(G_{\mathcal{J}\mathcal{J}'})=A_1\oplus A_3$ which implies $(\Gamma_{\mathcal{J}\mathcal{J}'})=D_1\oplus D_3$, with D_n a $n\times n$ matrix. Furthermore, by the formula for the inverse matrix $\tilde{D}_3(p^0=i\chi,\vec{p})$ (χ real) has the same structure as $\tilde{A}_3(p^0=i\chi,\vec{p})$ which in turn is the same as that of A_3. The dispersion curves $w_c(\vec{p})$ ($c=p,v$ with p, v referring to the pseudo-scalar and vector mesons, respectively) are given by

$$\begin{aligned} w_c(\vec{p}) &= -2\ln\kappa-6c_2\kappa^2+c_2\kappa^2\textstyle\sum_{j=1,2,3}2(1-\cos p^j)\\ &\quad+\kappa^4 r_c(\kappa,\vec{p})\,, \end{aligned}$$

$|r_c(\kappa,\vec{p})|=\mathcal{O}(1)$. $r_p(\kappa,\vec{p})$ is jointly analytic in κ and each p^j for $|\vec{p}|$ small, and is obtained by the auxiliary function method (see Ref. [15]). We still do not now if the dispersion curves are the same for the flavor singlet and octet. For the D_3 block, we have factorization of $\det_{3\times3}\tilde{D}_3(i\chi,\vec{p})=0$ using Cardano's formula for the roots of a cubic equation. However, due to analytical difficulties, we cannot apply the auxiliary function method, but Rouché's theorem (principle of the argument) can be applied to $\det_{3\times3}H(w,\kappa,\vec{p})=0$ to show that, for fixed $\vec{p}$, there are exactly three solutions.

V. EXTENSION OF THE SPECTRAL RESULTS TO ALL $\mathcal{H}_e$

Up to now, we have determined exactly the spectrum generated by vectors in $\mathcal{H}_{\bar{\mathcal{M}}}\subset\mathcal{H}_e$. But, it can happen that other fields can generate spectrum up to near the two-meson threshold ($\approx -4\ln\kappa$). As in Ref. [7], we use a correlation subtraction method to show that the eightfold way meson spectrum is the only spectrum in all $\mathcal{H}_e$, up to near the two-meson threshold of $\approx -4\ln\kappa$. For $L\in\mathcal{H}_e$ we have the spectral representation and F-K formula (with P_Ω the projection onto the vacuum state $\Omega\equiv 1$), $u^0\neq v^0$,

$$\Big((1-P_\Omega)L,\check{T}_0^{|v^0-u^0|-1}\check{\vec{T}}^{\vec{v}-\vec{u}}(1-P_\Omega)L\Big)_{\mathcal{H}}=\mathcal{G}(u,v)\,,$$

where, with $M=\Theta L$,

$$\begin{aligned}\mathcal{G}(u,v) &= \mathcal{G}_{ML}(u,v)\chi_{u^0\leq v^0}+\mathcal{G}_{ML}(u_t,v_t)\chi_{u^0>v^0}\\ &= \mathcal{G}_{ML}(u,v)\chi_{u^0\leq v^0}+\mathcal{G}^*_{LM}(u,v)\chi_{u^0>v^0},\end{aligned}$$

and we have used the notation $z_t=(-z^0,\vec{z})$ if $z=(z^0,\vec{z})$.

L may have contributions to the energy spectrum in the interval $(0,-(4-\epsilon)\ln\kappa)$ that arise from states not in $\mathcal{H}_{\bar{\mathcal{M}}}$. We show this is not the case by considering the decay of the subtracted function

$$\mathcal{F}=\mathcal{G}-\mathcal{P}\Lambda\mathcal{Q} \tag{17}$$

where the kernels of $\mathcal{P}$, Λ and $\mathcal{Q}$ are given by

$$\begin{aligned}\mathcal{P}(u,w) &= \mathcal{G}_{M\bar{\mathcal{M}}}(u,w)\chi_{u^0\leq w^0}+\mathcal{G}_{M\bar{\mathcal{M}}}(u_t,w_t)\chi_{u^0>w^0}\\ &= \mathcal{G}_{M\bar{\mathcal{M}}}(u,w)\chi_{u^0\leq w^0}+\mathcal{G}^*_{L\mathcal{M}}(u,w)\chi_{u^0>w^0},\end{aligned}$$

$$\begin{aligned}\mathcal{Q}(z,v) &= \mathcal{G}_{\mathcal{M}L}(z,v)\chi_{z^0\leq v^0}+\mathcal{G}_{\mathcal{M}L}(z_t,v_t)\chi_{z^0>v^0}\\ &= \mathcal{G}_{\mathcal{M}L}(z,v)\chi_{z^0\leq v^0}+\mathcal{G}^*_{\bar{\mathcal{M}}M}(z,v)\chi_{z^0>v^0},\end{aligned}$$

$$\begin{aligned}\mathcal{J}(w,z) &= \mathcal{G}_{\mathcal{M}\bar{\mathcal{M}}}(w,z)\chi_{w^0\leq z^0}+\mathcal{G}_{\mathcal{M}\bar{\mathcal{M}}}(w_t,z_t)\chi_{w^0>z^0}\\ &= \mathcal{G}_{\mathcal{M}\bar{\mathcal{M}}}(w,z)\chi_{w^0\leq z^0}+\mathcal{G}^*_{\bar{\mathcal{M}}\mathcal{M}}(w,z)\chi_{w^0>z^0}\end{aligned}$$

with $\Lambda(w,z)=\mathcal{J}^{-1}(w,z)$. The identities above are obtained using time reversal which gives

$$\mathcal{G}_{ML}(u_t,v_t)=\mathcal{G}^*_{LM}(u,v)\,,\ \mathcal{G}_{M\bar{\mathcal{M}}}(u_t,w_t)=\mathcal{G}^*_{L\mathcal{M}}(u,w),$$

$$\mathcal{G}_{\mathcal{M}L}(z_t,v_t)=\mathcal{G}^*_{\bar{\mathcal{M}}M}(z,v)\,,\ \mathcal{G}_{\mathcal{M}\bar{\mathcal{M}}}(w_t,z_t)=\mathcal{G}^*_{\bar{\mathcal{M}}\mathcal{M}}(w,z).$$

The motivation for the definitions of the kernels of $\mathcal{G}$ $\mathcal{P}$, Λ and $\mathcal{Q}$ is such that time reflected points give the same value for the $u^0<v^0$ and $u^0>v^0$ definitions.

The kernels of $\mathcal{P}$ and $\mathcal{Q}$ also have spectral representations for non-coincident temporal points given by

$$\Big(L,\check{T}_0^{|v^0-u^0|-1}\check{\vec{T}}^{\vec{v}-\vec{u}}\bar{\mathcal{M}}\Big)_{\mathcal{H}}=\mathcal{P}(u,v)\,,\ u^0\neq v^0$$

$$\Big(\bar{\mathcal{M}},\check{T}_0^{|v^0-u^0|-1}\check{\vec{T}}^{\vec{v}-\vec{u}}L\Big)_{\mathcal{H}}=\mathcal{Q}(u,v)\,,\ u^0\neq v^0.$$

We remark that in the two equations above we made use of $\langle\bar{\mathcal{M}}(u)\rangle=\langle\mathcal{M}(u)\rangle=0$ by parity symmetry. Using the hyperplane decoupling method we show below that $\mathcal{F}^{(r)}(u,v)=0,\ r=0,1,2,3$ for $|u^0-v^0|>2$ which implies that $\tilde{\mathcal{F}}(p)$ is analytic in p^0 in the strip $|\mathrm{Im}p^0|\leq-(4-\epsilon)\ln\kappa$. But $\tilde{\mathcal{F}}(p)=\tilde{\mathcal{G}}(p)-\tilde{\mathcal{P}}(p)\tilde{\Lambda}(p)\tilde{\mathcal{Q}}(p)$ so that possible singularities of $\tilde{\mathcal{G}}(p)$ in the strip are cancelled by those in the term $\tilde{\mathcal{P}}(p)\tilde{\Lambda}(p)\tilde{\mathcal{Q}}(p)$. From their spectral representations it is seen that $\tilde{\mathcal{P}}(p)$ and $\tilde{\mathcal{Q}}(p)$ only have singularities at the one-meson particle spectrum and the same holds for $\tilde{\mathcal{P}}(p)\tilde{\Lambda}(p)\tilde{\mathcal{Q}}(p)$ since $\tilde{\Lambda}(p)$ is analytic in the strip. Thus the singularities of $\tilde{\mathcal{G}}(p)$ and the spectrum generated by L in the interval $(0,-(4-\epsilon)\ln\kappa)$ are contained in the one-meson spectrum.

Expanding $\mathcal{F}$ in Eq. (17) in powers of κ_p we get the result

$$\begin{aligned}\mathcal{F} &= \mathcal{F}^{(0)}\kappa_p^0+\mathcal{F}^{(1)}\kappa_p+\mathcal{F}^{(2)}\kappa_p^2+\mathcal{O}(\kappa_p^3)\\ &= \big(\mathcal{G}^{(0)}-\mathcal{P}^{(0)}\Lambda^{(0)}\mathcal{Q}^{(0)}\big)\,\kappa_p^0\\ &\quad+\big(\mathcal{G}^{(1)}-\mathcal{P}^{(1)}\Lambda^{(0)}\mathcal{Q}^{(0)}-\mathcal{P}^{(0)}\Lambda^{(1)}\mathcal{Q}^{(0)}\\ &\quad-\mathcal{P}^{(0)}\Lambda^{(0)}\mathcal{Q}^{(1)}\big)\,\kappa_p+\big(\mathcal{G}^{(2)}-\mathcal{P}^{(2)}\Lambda^{(0)}\mathcal{Q}^{(0)}\\ &\quad-\mathcal{P}^{(0)}\Lambda^{(2)}\mathcal{Q}^{(0)}-\mathcal{P}^{(0)}\Lambda^{(0)}\mathcal{Q}^{(2)}-\mathcal{P}^{(1)}\Lambda^{(1)}\mathcal{Q}^{(0)}\\ &\quad-\mathcal{P}^{(1)}\Lambda^{(0)}\mathcal{Q}^{(1)}-\mathcal{P}^{(0)}\Lambda^{(1)}\mathcal{Q}^{(1)}\big)\,\kappa_p^2+\mathcal{O}(\kappa_p^3).\end{aligned}$$

That $\mathcal{F}^{(r)}(u,v)=0$ $(r=0,1,3)$ follows from gauge integration and imbalance of fermion fields appearing in the expectations. We now calculate the second derivative of $\mathcal{F}(u,v)$ for the time ordering $u^0\leq p<v^0$

$$\begin{aligned}\mathcal{F}^{(2)} &= \mathcal{G}^{(2)}-\mathcal{P}^{(0)}\Lambda^{(0)}\mathcal{Q}^{(2)}\\ &\quad-\mathcal{P}^{(0)}\Lambda^{(2)}\mathcal{Q}^{(0)}-\mathcal{P}^{(2)}\Lambda^{(0)}\mathcal{Q}^{(0)}\\ &= A_1+A_2+A_3+A_4.\end{aligned}$$

We will use in the sequel, for $r^0\leq p<s^0$ the following special cases of Eq.(9)

$$\mathcal{G}^{(2)}_{ML}(r,s)=\left[\mathcal{G}^{(0)}_{M\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\mathcal{M}L}\right](r,s),$$

$$\mathcal{G}^{(2)}_{M\bar{\mathcal{M}}}(r,s)=\left[\mathcal{G}^{(0)}_{M\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\mathcal{M}\bar{\mathcal{M}}}\right](r,s),$$

$$\mathcal{G}^{(2)}_{\mathcal{M}L}(r,s)=\left[\mathcal{G}^{(0)}_{\mathcal{M}\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\mathcal{M}L}\right](r,s),$$

$$\mathcal{G}^{(2)}_{\mathcal{M}\bar{\mathcal{M}}}(r,s)=\left[\mathcal{G}^{(0)}_{\mathcal{M}\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\mathcal{M}\bar{\mathcal{M}}}\right](r,s).$$

For the term A_2 we have

$$\begin{aligned}A_2 &= -\sum_{w^0,z^0\leq p}\mathcal{P}^{(0)}(u,w)\Lambda^{(0)}(w,z)\\ &\quad\times\left[\mathcal{G}^{(0)}_{\mathcal{M}\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\mathcal{M}L}\right](z,v)\\ &= -\sum_{\vec{w}}\mathcal{P}^{(0)}(u,(p,\vec{w}))\mathcal{G}^{(0)}_{\mathcal{M}L}((p+1,\vec{w}),v)\\ &= -\left[\mathcal{G}^{(0)}_{M\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\mathcal{M}L}\right](u,v)=-A_1\end{aligned}$$

where in the equation above we have extended the sum to all z using the support properties of $\Lambda^{(0)}$ and $\mathcal{G}^{(0)}$.

For the term A_4 we get similarly $A_4 = A_2$. Now we consider the term A_3:

$$A_3 = -\sum_{w^0 \le p\,,\, z^0 \ge p+1} \mathcal{P}^{(0)}(u,w)\Lambda^{(2)}(w,z)\mathcal{Q}^{(0)}(z,v).$$

But for, $w^0 \le p$, $z^0 \ge p+1$, $\Lambda^{(2)}(w,z) = -[\Lambda^{(0)}\mathcal{J}^{(2)}\Lambda^{(0)}](w,z)$ which is obtained taking the second derivative of the relation $\Lambda\mathcal{J} = 1$ and observing that $[\Lambda^{(0)}\mathcal{J}^{(1)}\ \Lambda^{(1)}](w,z) = [\Lambda^{(1)}\mathcal{J}^{(1)}\Lambda^{(0)}](w,z) = 0$ for $w^0 \le p$, $z^0 \ge p+1$. With these restrictions on sums we get

$$\mathcal{J}^{(2)}(x,y) = \mathcal{G}^{(2)}_{\mathcal{M}\bar{\mathcal{M}}}(x,y) = \left[\mathcal{G}^{(0)}_{\mathcal{M}\bar{\mathcal{M}}} \circ \mathcal{G}^{(0)}_{\mathcal{M}\bar{\mathcal{M}}}\right](x,y)$$

so that

$$\begin{aligned} A_3 \;=\; & \sum_{\substack{w^0, x^0 \le p\,, \\ z^0, y^0 \ge p+1}} \mathcal{P}^{(0)}(u,w)\Lambda^{(0)}(w,x) \\ & \times \left[\mathcal{G}^{(0)}_{\mathcal{M}\bar{\mathcal{M}}} \circ \mathcal{G}^{(0)}_{\mathcal{M}\bar{\mathcal{M}}}\right](x,y)\Lambda^{(0)}(y,z)\mathcal{Q}^{(0)}(z,v). \end{aligned}$$

Extending the sum to all x^0 and y^0 we get

$$\begin{aligned} A_3 \;&=\; \sum_{\vec{w}} \mathcal{P}^{(0)}(u,(p,\vec{w}))\mathcal{Q}^{(0)}((p+1,\vec{w}),v) \\ &=\; \left[\mathcal{G}^{(0)}_{M\bar{\mathcal{M}}} \circ \mathcal{G}^{(0)}_{\mathcal{M}L}\right](u,v) = A_1. \end{aligned}$$

Collecting the results above we get that $\mathcal{F}^{(2)}(u,v) = 0$ for $u^0 \le p < v^0$.

The treatment for the other time ordering, i.e. $u^0 \ge p > v^0$, is similar and we find that $\mathcal{F}^{(2)}(u,v) = 0$ for $u^0 > p+1$ and $v^0 < p$. Finally, taking into account the two time orderings we get $\mathcal{F}^{(2)}(u,v) = 0$ for the time separation $|u^0 - v^0| > 2$. The third derivative $\mathcal{F}^{(3)}(u,v) = 0$, $|u^0 - v^0| \ge 1$, is shown to be zero, similarly to the case $\mathcal{F}^{(1)}(u,v) = 0$, by imbalance of fermion fields.

VI. CONCLUDING REMARKS

In this work, concentrating our analysis in $\mathcal{H}_e$, we determined the low-lying energy-momentum spectrum exactly and showed the existence of all the eightfold way mesons (of asymptotic mass $-2\ln\kappa$) from dynamical first principles. It is shown that the masses admit representations which are jointly analytic in κ and β and, in particular, expansions in this parameters are controlled to all orders. For $\beta = 0$, we obtained a pseudo-scalar vector meson mass splitting giving by $2\kappa^4 + \mathcal{O}(\kappa^6)$ at $\beta = 0$ and, by analyticity, the splitting persists for $\beta > 0$, $\beta << \kappa$. Our result, combined with a similar one for baryons (of asymptotic mass $-3\ln\kappa$) (see Refs. [9, 10]), shows that the model exhibits confinement up to near the two-meson threshold ($\approx -4\ln\kappa$).

The difficulty encountered here in obtaining explicitly dispersion curves for the vector mesons disappears for the continuum if the vector fields transform under the Poincaré group where, the three identical dispersion curves are, of course, the relativistic ones.

In closing, we remark that the determination of the one meson spectrum opens the way to attack interesting open questions, such as, the existence of tetra-quarks and pentaquarks, for example meson-meson and meson-baryon bound states.

This work was supported by CNPq and FAPESP.

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