TOWARDS A MORE COMPLETE UNDERSTANDING OF NUCLEAR FORCES *

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We obtain an understanding of baryons and anti-baryons in the strong coupling regime of Euclidean lattice QCD. For a sufficiently small hopping parameter κ, we show that these particles arise as tightly bound, bound states of three (anti-)quarks associated with an upper gap in the energy-momentum spectrum with asymptotic mass of order $-3\ln\kappa$. The upper gap property holds in the full space of gauge invariant states, at least up to $-4\ln\kappa$ so that the corresponding dispersion curves are isolated. A spectral representation for the two-point baryon correlation and some symmetry properties for the model are also obtained. For simplicity, we consider the single flavor case in $2+1$ space-time dimensions and two-dimensional spin matrices. Our work here is a necessary intermediate step towards the analysis of e.g. two-baryon bound states and then to study how Yukawa forces are originated and can be understood from first principles. We end by outlining how this question can be attacked using our results on the energy-momentum spectrum of lattice quantum systems; and also we give partial results on how a mass splitting for baryons can be obtained from the lattice QCD dynamics.

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1. Introduction

It is a great honor to me to take part of this Festschrift volume to celebrate the eightieth anniversary of Professor Roberto Antonio Salmeron. I would like to point out that, in many aspects, Professor Salmeron has been very influential on me, since I first met him during my early days as an undergraduate in Physics at USP. Besides his life of entire dedication to any matter related to scientific research, his belief that high-level education is the right way for developing our country and his breathless work in this direction are overwhelming, constituting a good source of inspiration.

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Since particle physics is Professor Salmeron's main field of activity, I decided to center my contribution on one of the most natural questions that arise in quantum chromodynamics (QCD) to which, in my opinion, we unfortunately do not have yet a satisfactory answer, even after the many years since QCD was proposed as the model for strong interactions.

The existence and spectral properties of any quantum field theory is one of the fundamental problems in physics. In QCD, it is essential to establish on a rigorous basis the low energy-momentum spectrum of particles and their bound states. In particular, one needs to prove the existence of mesons and baryons, and their bound states.

Although some progress in this direction has been reached either considering QCD on the lattice [1, 2] or (less!) in the continuum, specifically, little is known regarding the baryon-baryon bound states (such as for the nuclear force) from first principles and there is still a gap to be bridged between QCD and the effective baryon, effective meson picture of nuclear forces through single and multiple boson exchange arising from the Yukawa interaction (see [3, 4]).

In this context, the determination and control of the baryonic mass spectrum is a necessary intermediate step towards the analysis of e.g. two-baryon bound states and then to study how nuclear Yukawa forces are originated and can be understood from first principles.

Spectral properties of this kind involve low-energies and one standard way to study them is to work within a lattice regularization of the continuum. We observe that long distance properties should not be affected too much by this procedure, that our understanding of confinement arises in this manner, and was indeed a reason for the introduction of lattice QCD by Wilson (see [5, 6] and [7, 8] for a review).

Here, I will be reporting on recent results [9] obtained in collaboration with Michael O'Carroll (ICMC-USP, São Carlos) and with Ricardo Schor (ICEx-UFMG, Belo Horizonte). They are thus be considered as co-authors of the main contents of these pages, for which I warmly thank them.

As our main result, we obtain an understanding of baryons from first principles. We show that they arise as tightly bound, bound states of three quarks. Their occurrence is manifested by the appearance of an isolated dispersion curve in the energy-momentum (e-m) spectrum. We also adopt a lattice approximation to the continuum QCD and obtain a qualitative picture of the spectrum from first principles in the strong coupling regime, corresponding to quark confinement. More precisely, we work with lattice QCD in the Euclidean formulation as given in ref. [6, 10], with small hopping parameter $0 < \kappa << 1$ and large glueball mass $1/g_0^2 << 1$) (see [11]). We make no other approximations so that, in this context, our results are exact. For the Hamiltonian approach to baryon masses we refer to ref. [1, 2], and to ref. [12] for the determination of masses for the meson sector, in the Euclidean approach.

Our choice of the Euclidean formulation is owed to the fact that it is so far the only formulation that can be mathematically justified, and for which the Osterwalder-Schrader reconstruction theorem guarantees how to recover the Minkowski space-time whilst preserving desired physical properties when their equivalent hold in the Euclidean space-time [13].

Finally, let us also observe that the development of the techniques we employ here, and which appear in our first paper on this subject (see [9]), generalizes the pioneering works of refs. [14], [15] and other several works involving quantum models on the lattice (see e.g. refs. [11, 16]). Recently, this formalism was improved and applied to a large variety of interesting lattice models, as in refs. [17], [18]. This is what makes our task feasible, in comparison e.g. to the problem of showing confinement in the continuum, which still seems to be out of reach. (We observe that the non-perturbative construction of the ultraviolet or continuum limit of the Euclidean four-dimensional Yang-Mills theory was accomplished in the past decade, in the long series of papers of ref. [19], from which QCD should follow with not much extra effort.)

2. The Model

To capture the essence of the mechanism of baryon formation while trying to avoid too much algebraic complexity, here we consider single flavor Fermi (quark) fields in $2 + 1$ space-time dimensions with two-dimensional Dirac (i.e. Pauli) spin matrices. Thus, there is *no* internal spin in our problem, and the two degrees of freedom are related to fermions and anti-fermions.

We show that there is a baryon particle and an anti-particle manifested by isolated dispersion curves in the e-m spectrum giving rise to an upper gap property in the e-m spectrum. Besides, we show that this is the *only* single-particle spectrum in the full space of states up to near the two-particle, two-baryon threshold. Hence, the upper gap persists in the whole Hilbert space. Also, analogous to the Källen-Lehman representation in quantum field theory, we obtain a spectral representation for the two-particle correlation function. This is one of the main ingredients which allows us to relate the singularities of its Fourier transform to the e-m spectrum via a Feynman-Kac formula. Furthermore, we establish and analyze some of the symmetries of the model.

It is important to stress that the upper gap property has not been established in the Hamiltonian formulation treatments.

The determination and control of the baryonic mass spectrum is an essential step towards an understanding of e.g. baryon-baryon bound states from first principles. Consequently, it is also a fundamental step towards a more complete understanding of the Yukawa (nuclear) force.

Let us define the gauge-matter model and show our results. The quantum mechanical Hilbert space and e-m operators are obtained by a standard construction from the thermodynamic limit of gauge-invariant correlations of a classical statistical mechanical model where the quark (fermionic) degrees of freedom are described by Grassmann variables (see [6, 10, 20]). Formally, the partition function of the model is given by

$$Z = \int e^{-S(\psi,\bar{\psi},g)}\, d\psi\, d\bar{\psi}\, d\mu(g) \ , \tag{1}$$

and, for $F(\bar{\psi},\psi,g)$, the normalized expectations are denoted by $\langle F\rangle$.

Calling 0 the temporal direction, the unit lattice is taken as $\mathbb{Z}^3_o$, where $u = (u^0,\vec{u}) = (u^0,u^1,u^2) \in \mathbb{Z}^3_o \equiv \mathbb{Z}_{1/2}\times\mathbb{Z}^2$, and $\mathbb{Z}_{1/2} = \{\pm 1/2, \pm 3/2, \ldots\}$. With this, in the continuum limit, the two-sided equal-time limits of Fermion correlations can be accommodated. At each site $u \in \mathbb{Z}^3_o$, we define independent Fermion Grassmann fields $\psi_{\alpha,a}(u)$, associated with quarks, and $\bar{\psi}_{\alpha,a}(u)$, associated with anti-quarks. Both fields carry a Dirac spin index $\alpha = 1,2 \equiv +,-$ and an $SU(3)$ fundamental group representation color index $a = 1,2,3$. For e^μ, denoting the unit vector for the $\mu = 0,1,2$ lattice direction, at each nearest neighbor positively oriented (according to each space-time axis) lattice bond $< u, u \pm e^\mu >$ there is associated an $SU(3)$ matrix $U(g_{u,u\pm e^\mu})$ which is parametrized by the gauge group element $g_{u,u\pm e^\mu}$ and satisfying $U(g_{u,u+e^\mu})^{-1} = U(g_{u+e^\mu,u})$, the later being associated with negatively oriented bonds. Also associated with each lattice oriented plaquette p there is a plaquette variable $\chi(U(g_p))$ where $U(g_p)$ is the orientation-ordered product of matrices of $SU(3)$ of the plaquette oriented bonds, and χ is the trace. The model action reads

$$\begin{aligned} S(\psi,\bar{\psi},g) = {} & \frac{\kappa}{2}\sum \bar{\psi}_{\alpha,a}(u)\Gamma^{\epsilon e^\mu}_{\alpha\beta}(g_{u,u+\epsilon e^\mu})_{ab}\psi_{\beta,b}(u+\epsilon e^\mu) \\ & + \sum_{u\in\mathbb{Z}^3_o} \bar{\psi}_{\alpha,a}(u) M_{\alpha\beta}\psi_{\beta,a}(u) - \frac{1}{g_0^2}\sum_p \chi(g_p) \ , \end{aligned} \tag{2}$$

where the first sum runs over $u \in \mathbb{Z}^3_o$, $\epsilon = \pm 1$ and $\mu = 0,1,2$. For the sake of a simple notation, we drop U from $U(g)$.

Our model is parametrized by a bare fermion mass-type parameter $m > 0$, a hopping parameter $\kappa > 0$ which measures the quark-gauge coupling, an inverse coupling $g_0 > 0$ describing the pure gauge interaction strength. The dependent variable M stands for $M = m + 2\kappa$. Also, within the family of actions of ref. [6], we have $\Gamma^{\pm e^\mu} = -1 \pm \gamma_\mu$, γ_μ being the hermitian traceless anti-commuting hermitian Pauli matrices σ_z, σ_x, σ_y, for $\mu = 0, 1, 2$, respectively. $d\mu(g)$ is the product measure over bonds of normalized $SU(3)$ Haar measures and the integrals over Grassmann fields are defined following ref. [20]. For a polynomial in the Grassmann variables with coefficients depending on the gauge variables, the fermionic integral is defined as the coefficient of the monomial of maximum degree, i.e. of $\prod_{u,\alpha,a} \bar\psi_{\alpha,a}(u)\psi_{\alpha,a}(u)$. In Eq. (1), $d\psi\, d\bar\psi$ means $\prod_{u,\alpha,a} d\psi_{\alpha,a}(u)\, d\bar\psi_{\alpha,a}(u)$ such that, with a normalization $N_1 = \langle 1 \rangle$, we have $\langle \psi_{\alpha,a}(x)\, \bar\psi_{\beta,b}(y) \rangle = (1/N_1) \int \psi_{\alpha,a}(x)\, \bar\psi_{\beta,b}(y)\, e^{-\sum_u \bar\psi_{\alpha,a}(u) M_{\alpha\beta} \psi_{\beta,a}(u)}\, d\psi\, d\bar\psi = M^{-1}_{\alpha,\beta} \delta_{ab} \delta(x-y)$, with a Kronecker delta for space-time coordinates. With our restrictions on the parameters, there is a quantum mechanical Hilbert space of physical states (see below), for $\kappa > 0$; and the condition $m > 0$ guarantees that the one-particle free Fermion dispersion curve increases separately in each positive momentum component.

For small enough couplings κ and g_0^{-2}, using polymer expansions (see [6, 21]), we can show that the thermodynamic (infinite-volume) limit of correlations exists and that truncated correlations have exponential tree decay. The limiting correlation functions are translation invariant on the lattice and the correlations extend to analytic functions in the coupling parameters.

To continue, we recall the definition of the quantum mechanical Hilbert space $\mathcal{H}$ and the e-m operators starting from gauge invariant correlation functions, with support restricted to $u^0 = 1/2$. Let $T_0^{x^0}$, $T_i^{x^i}$, $i = 1, 2$, denote translation of the functions of Grassmann and gauge variables by $x^0 \geq 0$, $x \in \mathbb{Z}^3$; and for F and G depending only on coordinates, with $u^0 = 1/2$, we have the Feynman-Kac (F-K) type formula

$$(G, T_0^{x^0} T_1^{x^1} T_2^{x^2} F)_{\mathcal{H}} = \langle [T_0^{x^0} T_1^{x^1} T_2^{x^2} F] \Theta G \rangle \,,$$

where Θ is an anti-linear operator which involves time reflection.

Following ref. [6], with the usual sum convention, the action of Θ on single fields is given by $\Theta \bar\psi_{\alpha,a}(u) = (\gamma_0)_{\alpha\beta} \psi_{\beta,a}(tu)$ and $\Theta \psi_{\alpha,a}(u) = \bar\psi_{\beta,a}(tu)(\gamma_0)_{\beta\alpha}$, where $t(u^0, \vec{u}) = (-u^0, \vec{u})$, for A and B monomials, $\Theta(AB) = \Theta(B)\Theta(A)$; and for a function of the gauge fields $\Theta f(\{g_{uv}\}) =$

$f^*(\{g_{(tu)(tv)}\})$, $u, v \in \mathbb{Z}_o^3$, where $*$ means complex conjugation. Θ extends anti-linearly to the algebra. Our notation does not make a distinction between Grassmann, gauge variables and their associated Hilbert space vectors in our notation. As linear operators in $\mathcal{H}$, T_μ, $\mu = 0, 1, 2$, are mutually commuting; T_0 is self-adjoint, with $-1 \leq T_0 \leq 1$, and $T_{j=1,2}$ are unitary, so that we write $T_j = e^{iP^j}$ and $\vec{P} = (P^1, P^2)$ is the self-adjoint momentum operator, with spectral points $\vec{p} \in \mathbf{T}^2 \equiv (-\pi, \pi]^2$. It follows from $T_0^2 \geq 0$ that we can define the energy operator $H \geq 0$ by $T_0^2 = e^{-2H}$. More specifically, the positivity condition $\langle F\Theta F\rangle \geq 0$ is established in ref. [6] but there may be nonzero F's such that $\langle F\Theta F\rangle = 0$. The collection of such F's is denoted by $\mathcal{N}$. Thus, a pre-Hilbert space $\mathcal{H}'$ can be constructed from the inner product $\langle G\Theta F\rangle$. The physical space $\mathcal{H}$ is defined as the completion of $\mathcal{H}'/\mathcal{N}$.

3. Main Results

Now, we precise our results. For this, we restrict ourselves to the subspace $\mathcal{H}_o \subset \mathcal{H}$ generated by an odd number of $\hat{\psi} = \bar{\psi}$ or ψ, for $g_0^{-2} << \kappa$. For the pure gauge case and small g_0^{-2}, the low-lying glueball spectrum is found in ref. [11]. For large g_0, the glueball mass is $\approx 8 \ln g_0$. For the even subspace, quark-anti-quark scalar mesons are treated in ref. [12].

To show the existence of baryon particles, let us consider the subspace $\mathcal{H}_3 \subset \mathcal{H}$ generated by vectors associated with Grassmann gauge-invariant baryon-like fields given by, with ϵ_{abc} denoting the Levi-Civita symbol,

$$\hat{\phi}_\pm\left(u^0 = 1/2, \vec{u}\right) = \tfrac{1}{6}\epsilon_{abc}\hat{\psi}_{a\pm}\hat{\psi}_{b\pm}\hat{\psi}_{c\pm} = \hat{\psi}_{1\pm}\hat{\psi}_{2\pm}\hat{\psi}_{3\pm}\,,$$

with either three ψ's or three $\bar{\psi}$'s. For small enough $\kappa >> g_0^{-2} > 0$, and for energies less than $-(4-\epsilon)\ln\kappa$, $0 < \epsilon << 1$, we show that in $\mathcal{H}_3$ the only e-m spectrum consists of dispersion curves $w_\pm(\vec{p})$, where

$$\begin{aligned} w_\pm(\vec{p}) &= 3\ln\tfrac{M}{2\kappa} + \ln[1 - 2\tfrac{\kappa^3}{M^3}(\cos p^1 + \cos p^2)] + \mathcal{O}(\kappa^4) \\ &= m_{\pm,\kappa} + \tfrac{\kappa^3}{M^3}|\vec{p}\,|^2 + \mathcal{O}(\kappa^4) \quad , \quad |\vec{p}\,| << 1\,; \end{aligned}$$

with $m_{\pm,\kappa} \equiv w_\pm(\vec{0})$ being the $\hat{\phi}_\pm$ baryon masses.

Also, adapting the methods of refs. [21, 11], it can be shown that $w_\pm(\vec{p}) + 3\ln\kappa$ is real analytic in κ, for $|\kappa|$ small, and in each p^i, for $|\mathcal{I}\mathrm{m}\ p^i|$ small. Furthermore, the dispersion curves $w_\pm(\vec{p})$ are increasing in each p^i, and convex for small $|\vec{p}|$. It is important to remark that, in the case of free quarks, the system kinetic energy for the three-particle threshold has an $\mathcal{O}(\kappa)$ coefficient, rather than a κ^3 coefficient. We associate the curve

$w_-(\vec{p})$ with $\bar{\phi}_-$ and call it a baryon field. Similarly $w_+(\vec{p})$ is associated with the anti-baryon field ϕ_+. This terminology is justified. Using a charge conjugation symmetry given below, we show the two dispersion curves are indeed identical [9].

Associated with the baryon particle and anti-particle there are normalized two-point functions

$$\begin{aligned} G_-(u,v) &= \langle \phi_-(u)\bar{\phi}_-(v)\rangle\, \chi_{u^0\leq v^0} - \langle \bar{\phi}_-(u)\phi_-(v)\rangle\, \chi_{u^0>v^0} = G_-(u-v)\ , \\ G_+(u,v) &= \langle \bar{\phi}_+(u)\phi_+(v)\rangle\, \chi_{u^0\leq v^0} - \langle \phi_+(u)\bar{\phi}_+(v)\rangle\, \chi_{u^0>v^0} = G_+(u-v)\ ; \end{aligned}$$

where here χ denotes the characteristic or indicator function. We note that by parity symmetry defined later on, the $u^0 = v^0$ extensions of the $u^0 > v^0$ parts agree with the corresponding $u^0 = v^0$ definitions.

Dropping the subscript $-$, we now restrict our discussion to G_-, the analysis of G_+ being similar. From the F-K formula

$$\begin{aligned} G(x) = &-(\bar{\phi}_-(1/2,\vec{0}), T_0^{|x^0|-1} e^{i\vec{P}.\vec{x}} \bar{\phi}_-(1/2,\vec{0}))_{\mathcal{H}}\ \chi_{x^0>0} \\ &-(\bar{\phi}_-(1/2,\vec{0}), T_0^{|x^0|-1} e^{-i\vec{P}.\vec{x}} \bar{\phi}_-(1/2,\vec{0}))_{\mathcal{H}}\ \chi_{x^0<0}\ , \end{aligned}$$

letting $\tilde{G}(p)$ denote the Fourier transform of $G(x \equiv u - v)$, taking the Fourier transform of G, and using the spectral representations for T_0, T_1 and T_2, we obtain the following spectral representation

$$\begin{aligned} \tilde{G}(p) = \tilde{G}(\vec{p}) + (2\pi)^2 \int_{-1}^{1}\int_{\mathbf{T}^2} &\frac{2(\cos p^0 - \lambda^0)}{1+(\lambda^0)^2 - 2\lambda^0\cos p^0}\ \times \\ &\delta(\vec{p}-\vec{\lambda}) d_{\lambda^0} d_{\vec{\lambda}} (\bar{\phi}_-(1/2,\vec{0}), \mathcal{E}(\lambda^0)\mathcal{F}(\vec{\lambda})\phi_-(1/2,\vec{0}))_{\mathcal{H}}\ , \end{aligned}$$

where $\tilde{G}(\vec{p}) = \sum_{\vec{x}\in\mathbb{Z}^2} e^{-i\vec{p}.\vec{x}} G(x^0 = 0, \vec{x})$ and $\mathcal{F}(\vec{\lambda}) = \mathcal{F}_1(\lambda^1)\mathcal{F}_2(\lambda^2)$, and $\mathcal{E}(\lambda^0)$ (respectively, $\mathcal{F}_i(\lambda^i)$,$i = 1,2$) is the spectral family for the operator T_0 (respectively P^i).

The existence of an isolated dispersion curve in the e-m spectrum occurs as a singularity of $\tilde{G}(p)$ on the imaginary p^0 axis, for fixed $\vec{p}$. (Our analysis excludes a singularity for $p^0 = i\pi + q^0$, q^0 real). Using a decoupling of hyperplane method (see [21, 11]), more precise bounds on $G(x)$ and on its convolution inverse $\Gamma(x)$ can be obtained. $|G(x)|$ is bounded by $\mathrm{const}\, \mathrm{e}^{-3|\ln(\kappa)\, \mathrm{x}^0| - 2|\ln(\kappa)\, \vec{\mathrm{x}}|}$ and hence $\tilde{G}(p)$ is analytic in p^0 in $|\mathcal{I}\mathrm{m}\ p^0| < 3|\ln\kappa|$. The decay of the convolution inverse $\Gamma(x)$ is faster and bounded by $\mathrm{const}\, \mathrm{e}^{-4|\ln(\kappa)\, \mathrm{x}^0| - 2|\ln(\kappa)\, \vec{\mathrm{x}}|}$. By the Paley-Wiener theorem [22], this implies that $\tilde{\Gamma}(p)$ is analytic in the larger strip $|\mathcal{I}\mathrm{m}\ p^0| < -4\ln\kappa$ and, consequently, that $\tilde{\Gamma}(p)^{-1}$ provides a meromorphic extension of $\tilde{G}(p)$ to this region. Besides, using the short distance behavior of $G(x)$, obtained by expanding perturbatively in κ, namely $G(0) = -c^3 + \mathcal{O}(\kappa^8)$,

$G(\pm e^0) = -8(\kappa/2)^3 c^6 + \mathcal{O}(\kappa^5)$, $G(\pm e^i) = -(\kappa/2)^3 c^6 + \mathcal{O}(\kappa^5)$, where $c = 2/M$, we find $\tilde{G}(p) = -c^3[1+2(\kappa/2)^3 c^3(8\cos p^0 + \cos p^1 + \cos p^2)] + \mathcal{O}(\kappa^5)$ and $\tilde{\Gamma}(p) = -c^{-3}[1 - 2(\kappa/M)^3(8\cos p^0 + \cos p^1 + \cos p^2)] + \mathcal{O}(\kappa^5)$. The dispersion curve $w(\vec{p})$ corresponds to the solution of $\tilde{\Gamma}(p^0 = -iw(\vec{p}), \vec{p}) = 0$.

With this, we show our spectral results considering only the subspace $\mathcal{H}_3 \subset \mathcal{H}_o$. To extend the spectral results to all of $\mathcal{H}_o$, as our main tool, we adapt the subtraction method of ref. [11].

We will now establish lattice charge conjugation and parity symmetries.

4. Analysis of Some Symmetries

The equality of the dispersion curves $w_\pm(\vec{p})$ follows from the former and the parity symmetry is used to define an intrinsic (anti-)baryon parity. For the gauge-Grassmmanian field algebra, we define the charge conjugation transformation C by

$$C\psi_{\alpha,a}(u) = \bar{\psi}_{\beta,a}(u)\,(\gamma_2)_{\beta\alpha}\ ;\ C\bar{\psi}_{\alpha,a}(u) = (\gamma_2)_{\alpha\beta}\,\psi_{\beta,a}(u);$$
$$Cf(\{g_{uv}\}) = f(\{g^*_{uv}\})\,;\ C(AB) = C(B)C(A).$$

where A and B are Grassmannian monomials. The transformation is extended by linearity to a general element.

The action of Eq. (2) is invariant under C, such as $CS = S$, and $\langle CF\rangle_0 = \langle F\rangle_0$, where $\langle\,\cdot\,\rangle_0$ denotes the expectation with the first term (hopping term) in the action S set equal to zero. From these two properties, it follows that $\langle CF\rangle = \langle F\rangle$. In showing $\langle CF\rangle_0 = \langle F\rangle_0$, we use the explicit formula for the integration of a class (gauge invariant) function with the $SU(3)$ Haar measure (see e.g. [23]). Since $C\phi_-(u) = -i\bar{\phi}_+(u)$ and $C\bar{\phi}_-(u) = i\phi_+(u)$, we have $\langle\phi_-(u)\bar{\phi}_-(v)\rangle = -\langle\bar{\phi}_+(u)\phi_+(v)\rangle$. From the F-K formulae, for $x^0 \geq 0$, $(\phi_-, e^{-Hx^0} e^{i\vec{P}.\vec{x}}\phi_-)_\mathcal{H} = -\langle\phi_-(x^0+1/2,\vec{x})\bar{\phi}_-(-1/2,\vec{0})\rangle$, $(\bar{\phi}_+, e^{-Hx^0} e^{i\vec{P}.\vec{x}}\bar{\phi}_+)_\mathcal{H} = \langle\bar{\phi}_+(x^0+1/2,\vec{x})\phi_+(-1/2,\vec{0})\rangle$, so that, by the equality of the right hand sides; such as the equality of moments of the spectral measure, we have the equality $(\phi_-, \mathcal{E}(\lambda^0)\mathcal{F}(\vec{\lambda})\phi_-)_\mathcal{H} = (\bar{\phi}_+, \mathcal{E}(\lambda^0)\mathcal{F}(\vec{\lambda})\bar{\phi}_+)_\mathcal{H}$ which implies the equality of the spectrum; in particular, the dispersion curves are identical. Furthermore, from $\langle(CF)(\theta CF)e^{-S}\rangle_0 = \langle F\theta F e^{-S}\rangle_0$, C can be lifted to $\mathcal{H}$ as a unitary operator. Thus, if $F \in \mathcal{N}$, also $CF \in \mathcal{N}$.

Next, we consider the parity symmetry. We define the parity transformation P, with $\mathcal{P}(u^0,\vec{u}) = (u^0,-\vec{u})$, by ($A$ and B are Grassmann monomials)

$$P\psi_{\alpha,a}(u) = (\gamma_0)_{\alpha\beta}\,\psi_{\beta,a}(\mathcal{P}u)\ ,\ P\bar{\psi}_{\alpha,a}(u) = \bar{\psi}_{\beta,a}(\mathcal{P}u)\,(\gamma_0)_{\beta\alpha}\ ,$$
$$Pf(\{g_{uv}\}) = f(\{g_{\mathcal{P}u\mathcal{P}v}\})\quad,\qquad P(AB) = P(A)P(B)\ ,$$

and extend it by linearity. $PS = S$ and $\langle PF\rangle_0 = \langle F\rangle_0$ which implies $\langle PF\rangle = \langle F\rangle$. As $\langle PF\Theta PF\rangle = 0$ if $\langle F\Theta F\rangle = 0$, P can be lifted to an operator acting in $\mathcal{H}$ and P commutes with the Hamiltonian H. In particular, $P\bar{\phi}_-(u) = -\bar{\phi}_-(u^0, -\vec{u})$ and $P\phi_+(u) = \phi_+(u^0, -\vec{u})$, so that the improper states, with $u^0 = 1/2$, $\bar{\Phi}_-(\vec{p}) \equiv \sum_{\vec{u}\in\mathbb{Z}^2} e^{-i\vec{p}.\vec{u}}\bar{\phi}_-(u)$ and $\Phi_+(\vec{p}) \equiv \sum_{\vec{u}\in\mathbb{Z}^2} e^{-i\vec{p}.\vec{u}}\phi_+(u)$ satisfy $P\bar{\Phi}_-(\vec{0}) = -\bar{\Phi}_-(\vec{0})$ and $P\Phi_+(\vec{0}) = \Phi_+(\vec{0})$, i.e. $\bar{\Phi}_-(\vec{p} = \vec{0})$ and $\Phi_+(\vec{p} = \vec{0})$ have eigenvalues $+1$ and -1, respectively, and we identify these eigenvalues as the intrinsic parities.

Another consequence of the invariance of expectations under the parity transformation is that $w(\vec{p}) = w(-\vec{p})$; this follows since $\langle \phi_-(x)\bar{\phi}_-(y)\rangle = \langle P[\phi_-(x)\bar{\phi}_-(y)]\rangle = \langle \phi_-(x^0, -\vec{x})\bar{\phi}_-(y^0, -\vec{y})\rangle$ which implies $\tilde{G}(p^0, -\vec{p}) = \tilde{G}(p^0, \vec{p})$ and $\tilde{\Gamma}(p^0, -\vec{p}) = \tilde{\Gamma}(p^0, \vec{p})$. A spatial rotation by $\pi/2$ is also a symmetry and is defined by a transformation R similar to that of parity replacing P by R and $\mathcal{P}$ by $\mathcal{R}$, where $\mathcal{R}(x^0, x^1, x^2) = (x^0, x^2, -x^1)$ and γ_0 by either $(1 + \gamma_0)/2 + i(1 - \gamma_0)/2$, or with $-i$. With these choices, $R^2 = P$.

5. More on Results

Now, we show how the interplay of the hyperplane decoupling method and gauge invariance not only reveals the lowest e-m excitations but also shows that the excitations are particles, i.e. have isolated dispersion curves. Consider the correlation function of two functions $H(x)$ and $L(y)$ localized at x and y, respectively, with $x^0 < y^0$. We assume that $\langle H(x)\rangle = 0 = \langle L(y)\rangle$. In the hyperplane decoupling method (see [21, 11]), the parameter κ in the action is replaced by the complex parameter κ_p for all bonds connecting the hyperplane $u^0 = p$, p being a half-integer, and $u^0 = p + 1$. We consider the resulting $\langle H(x)L(y)\rangle$ and its derivatives at $\kappa_p = 0$. Now, with $G(x, y, \kappa_p) \equiv \langle H(x)L(y)\rangle$, $G(x, y, \kappa_p = 0) = 0$ and if the first non-vanishing derivative, say the derivative of order k, is non-vanishing, and if we can find $L(x)$ and $H(y)$ such that, at $\kappa_p = 0$,

$$\frac{\partial^k}{\partial\kappa_p^k}\, G(x, y, 0) \propto \sum_{\vec{z}\in\mathbb{Z}^2} G(x, (p, \vec{z}))G((p + 1, \vec{z}), y)\,,$$

then, for the k-th derivative of minus the convolution inverse Γ' ($\Gamma' G = -1$), given by (Leibniz's rule) $\frac{\partial^k}{\partial\kappa_p^k}\,\Gamma'|_{\kappa_p=0} = \sum_{n=0}^{k-1}\begin{pmatrix} k \\ n\end{pmatrix}\Gamma'\frac{\partial^{k-n}G}{\partial\kappa_p^{k-n}}\,\frac{\partial^n\Gamma'}{\partial\kappa_p^n}\,|_{\kappa_p=0}$, only the $\ell = k$, $n = 0$ term contributes. Using the above structure of $\frac{\partial^k}{\partial\kappa_p^k}\, G\,|_{\kappa_p=0}$, we conclude that $\frac{\partial^k}{\partial\kappa_p^k}\,\Gamma'\,|_{\kappa_p=0} = 0$, $x^0 \leq p < y^0 - 1$. This is the important ingredient that leads to the faster decay rate for Γ, as compared to G, i.e. $\mathrm{const}\, \mathrm{e}^{-4|\mathrm{x}^0 \ln\kappa|}$. The decay is obtained by repeating the argument for each hyperplane with $x^0 \leq p \leq y^0 - 1$, using joint analyticity in κ_p, $x^0 \leq p < y^0$, and Cauchy bounds for the κ_p derivatives (see [21, 11]).

In our case, the coefficient of κ_p^n in the numerator of G is of the form

$$\frac{1}{n!} \int H(x) \left(\sum_{\vec{z} \in \mathbb{Z}^2} \ldots \right)^n L(y) e^{-S(\bar{\psi}, \psi, g)} \, d\psi d\bar{\psi} d\mu(g) \left.\right|_{\kappa_p = 0} ,$$

and we now perform the integral over the gauge field for the bonds between the hyperplanes. For $n = 1$, by the Peter-Weyl (see [23]) orthogonality relations the integration gives zero. For $n = 2$, if H and L are in the odd subspace of $\mathcal{H}$, then the Fermi integration gives zero. For $n = 3$, the gauge integral is zero unless the three bonds coincide. In the case of free Fermi fields, there is no such restriction. For coincident bonds a three-fold tensor product, where each factor is either U or $\bar{U}$ (with the bar denoting complex conjugate) occurs. Using the decomposition of tensor products of $SU(3)$, namely $3 \times 3 \times 3 = 10 + 8 + 8 + 1$, $3 \times 3 \times \bar{3} = 15 + \bar{6} + 3 + 3$, and their complex conjugates, only the identity representation contributes and the third derivative at $\kappa_p = 0$ can be written as

$$\begin{aligned} \tfrac{1}{3!} \tfrac{\partial^3}{\partial \kappa_p^3} \langle H(x) L(y) \rangle = -8 \textstyle\sum_{\vec{z} \in \mathbb{Z}^2} \, \big[& \langle \bar{\phi}_+(p+1, \vec{z}) L(y) \rangle \, \langle H(x) \phi_+(p, \vec{z}) \rangle \\ & - \langle H(x) \bar{\phi}_-(p, \vec{z}) \rangle \, \langle \phi_-(p+1, \vec{z}) L(y) \rangle \, \big] \; . \end{aligned}$$

Choosing $H(x) = \phi_-(x)$ and $L(y) = \bar{\phi}_-(y)$, we have, at $\kappa_p = 0$,

$$\frac{1}{3!} \frac{\partial^3 G_-}{\partial \kappa_p^3}(x, y, 0) = 8 \sum_{\vec{z} \in \mathbb{Z}^2} G_- \left(x, (p, \vec{z}) \right) G_- \left((p+1, \vec{z}), y \right) \, ,$$

such as, the required derivative structure. Similar considerations apply for G_+, choosing $H(x) = \bar{\phi}_+(x)$ and $L(y) = \phi_+(y)$, and $x^0 > p \geq y^0$.

Next, we show that the only e-m spectrum in all of $\mathcal{H}_o$, up to the energy value $-(4 - \epsilon) \ln \kappa$, is the isolated baryon dispersion curve $w(\vec{p}) \equiv w_\pm(\vec{p})$. It is enough to show that for $L \in \mathcal{H}_o$ with finite spatial support and time support at $u^0 = 1/2$, the only possible e-m contribution, up to energies $-(4 - \epsilon) \ln \kappa$, in $(L, e^{-Hx^0} e^{i\vec{P}.\vec{x}} L)_{\mathcal{H}}$ comes from $w(\vec{p})$. By the F-K formula $(L, e^{-Hx^0} e^{i\vec{P}.\vec{x}} L)_{\mathcal{H}} = \langle L(x) \hat{L} \rangle$, $x^0 \geq 0$, where $\hat{L} \equiv \Theta L$, and in general we write $K(x)$ for the translation of K by x. Letting $G(x) = \langle L(x^0 - 1, \vec{x}) \hat{L} \rangle$, $x \in \mathbb{Z}^3$, and using the F-K formula, the Fourier transform $\tilde{G}(p)$ of $G(x)$, has the spectral representation $\tilde{G}(p) = \tilde{G}(\vec{p}) + (2\pi)^{d-1} \int_{-1}^{1} \int_{\mathbf{T}^2} \frac{1}{e^{ip^0} - \lambda^0} \delta(\vec{p} - \vec{\lambda}) d_{\lambda^0} d_{\vec{\lambda}} (\hat{L}, \mathcal{E}(\lambda^0) \mathcal{F}(\vec{\lambda}) \hat{L})_{\mathcal{H}} - (2\pi)^{d-1} \int_{-1}^{1} \int_{\mathbf{T}^2} \frac{1}{e^{-ip^0} - \lambda^0} \delta(\vec{p} + \vec{\lambda}) \, d_{\lambda^0} d_{\vec{\lambda}} (L, \mathcal{E}(\lambda^0) \mathcal{F}(\vec{\lambda}) L)_{\mathcal{H}}$, where $\tilde{G}(\vec{p}) = \sum_{\vec{x} \in \mathbb{Z}^2} e^{-i\vec{p}.\vec{x}} G(x^0 = 0, \vec{x})$. We define, with $G(x, y) = G(x - y)$,

$F(x,y) = G(x,y) + \sum_{u,v\in\mathbb{Z}^3}\langle L(x)\Phi(u)\rangle\Gamma(u,v)\langle\chi(v)\hat{L}(y)\rangle$, where $\Phi(u) = (\phi_+(u),\bar{\phi}_-(u))_{\mathcal{H}}$ and $\chi(v) = (\bar{\phi}_+(v),-\phi_-(v))_{\mathcal{H}}$ have two components; $\Gamma(u,v)$ is a 2×2 matrix and we suppress the sum over components, Γ is minus the convolution inverse of M, where $M(x,y) = \mathrm{diag}\left[(\bar{\phi}_+(x),\phi_+(y))_{\mathcal{H}} - (\phi_-(x),\bar{\phi}_-(y))_{\mathcal{H}}\right]$. The second term in $F(x,y)$ is designed so that we have: *i)* that the possible singularities, in $|\mathcal{I}\mathrm{m}\ p^0| < -(4-\epsilon)\ln\kappa$, in $\tilde{G}(p)$ are cancelled; *ii)* the only possible singularities in the Fourier transform come from $w(\vec{p})$.

By the hyperplane decoupling method applied to $F(x,y)$, $x^0 \le p < y^0$, it is seen that $\partial^k F(x,y)/\ \partial\kappa_p^k = 0$, for $k = 0,1,2,3$, so that $|F(x,y)| \le$ const $e^{-(4-\epsilon)|\ln(\kappa)\,(x^0-y^0)|}$, which implies analyticity of $\tilde{F}(p)$, the Fourier transform of $F(x,0)$, in $|\mathcal{I}\mathrm{m}\ p^0| < -(4-\epsilon)\ln\kappa$. For $\tilde{F}(p)$, we have $\tilde{F}(p) = \tilde{G}(p) + \tilde{H}(p)\tilde{\Gamma}(p)\tilde{J}(p)$, where $\tilde{H}(p)$, $\tilde{\Gamma}(p)$ and $\tilde{J}(p)$ are the Fourier transforms of $\langle L(x)\Phi(0)\rangle$, $\Gamma(x,0)$ and $\langle\chi(x)\hat{L}(0)\rangle$, respectively. $\tilde{H}(p)$ and $\tilde{J}(p)$ have spectral representations which are similar to the one for $\tilde{G}(p)$ given above, with the only possible singularities given by $w(\vec{p})$ in $|\mathcal{I}\mathrm{m}\ p^0| < -(4-\epsilon)\ln\kappa$ and $\tilde{\Gamma}(p)$ is analytic in this region. It follows that the only possible singularities of $\tilde{G}(p)$ are those arising from $w(\vec{p})$. This completes our argument.

6. Partial Results and Speculations

There are two natural ways to continue the analysis of the problem given above. One is the question to go up in the e-m spectrum to accomplish the two-particle (two-baryon) bound-state analysis. The other is to see how, and under which conditions, we can have mass splitting in the baryon particle sector. This has been done for the mesonic particles in ref. [12].

These two questions are under consideration in refs. [24, 25], and we go through some speculations based on what our partial results seem to support, hoping they will become substantial in the near future.

Let us first focus on the baryon-baryon bound state problem, outlining how it can be attacked. Following ref. [17], after establishing a Feynman-Kac formula for the four-point baryon type function, a test for the presence of two-particle bound states can be done by studying the analyticity structure of the Bethe-Salpeter (B-S) equation. In operator form this equation is $D = D_0 + D_0KD$ and defines K. In terms of kernels, with $x_1^0 = x_2^0$, $x_3^0 = x_4^0$, we obtain

$$D(x_1,x_2,x_3,x_4) = D_0(x_1,x_2,x_3,x_4) + \int D_0(x_1,x_2,y_1,y_2)K(y_1,y_2,y_3,y_4) \times\ D(y_3,y_4,x_3,x_4)\delta(y_1^0-y_2^0)\,\delta(y_3^0-y_4^0)\,dy_1dy_2dy_3dy_4\,,$$

where $D_0(x_1,x_2,x_3,x_4) = -\langle\phi_-(x_1)\bar\phi_-(x_3)\rangle\langle\phi_-(x_2)\bar\phi_-(x_4)\rangle + \langle\phi_-(x_1)\bar\phi_-(x_4)\rangle\langle\phi_-(x_2)\bar\phi_-(x_3)\rangle$, $D(x_1,x_2,x_3,x_4) = \langle\phi_-(x_1)\phi_-(x_2)\bar\phi_-(x_3)\bar\phi_-(x_4)\rangle$ is truncated for the x_1,x_2 and x_3,x_4 clusters, and we use a continuum notation for sums over lattice points. Of course, we can also write similar expressions involving the $+$ fields $\bar\phi_+$ and ϕ_+.

The quantities D, D_0 and K are to be taken as matrix operators acting on $\ell_2^a(\mathcal{A})$, the anti-symmetric subspace of $\ell_2(\mathcal{A})$, where $\mathcal{A} = \big\{(x_1,x_2)\in\mathbb{Z}^3\times\mathbb{Z}^3/x_1^0 = x_2^0\big\}$. Formally, we have $K = D_0^{-1} - D^{-1}$, and we remark that after taking into account the cancellation in D^{-1} due to its Gaussian-like (Wick theorem satisfied but with full propagators!) equivalent contribution D_0^{-1} the B-S equation puts into evidence a one-particle irreducible structure for K connecting the two x_1,x_2 and x_3,x_4. To see this in our formalism, we use the hyperplane decoupling expansion method (see [21, 11]), as before, and check the vanishing of six derivatives with respect to κ_p, allowing us to go up to near the two-particle threshold which is approximately of order $-6\ln\kappa$.

Using translation invariance on the lattice to pass first to difference coordinates, we can re-write the B-S equation in terms of the lattice relative coordinates $\xi = x_2 - x_1$, $\eta = x_4 - x_3$ and $\tau = x_3 - x_2$. In terms of $(\vec\xi,\vec\eta,\tau)$, and taking the Fourier transform in τ only, the B-S equation becomes

$$\hat D(\vec\xi,\vec\eta,k) = \hat D_0(\vec\xi,\vec\eta,k) + \int \hat D_0(\vec\xi,\vec\xi',k)\hat K(-\vec\xi',-\vec\eta',k)\hat D(\vec\eta',\vec\eta,k)\,d\vec\xi' d\vec\eta'\,.$$

$\hat K(-\vec\xi',-\vec\eta',k)$ acts as an energy-dependent non-local potential in the non-relativistic lattice Schrödinger operator analogy.

The bound state analysis can then be done in a first step by considering L, the ladder approximation to K. This corresponds to a local potential approximation, which is determined by keeping only the diagonal parts D_d of D and $D_{0,d}$ of D_0, respectively, up to $\mathcal{O}(\kappa^n)$, for some n. Once the ladder approximation analysis is done, in refs. [15, 17] it is shown how perturbations to the ladder approximation can be controlled to go beyond the ladder approximation and establish spectral results for the full model.

The analysis of the problem in the ladder approximation is under consideration in ref. [24] and there are forthcoming results on the way.

We now turn to the mass splitting problem which is being considered in ref. [25]. For this, in order to see the complete nontrivial structure introduced by rotations and spin, we consider the 3+1 dimensional problem, and take four 4×4 Euclidean Dirac matrices $\gamma^0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$; $\vec\gamma = \begin{pmatrix}0 & -i\vec\sigma\\ i\vec\sigma & 0\end{pmatrix}$, in the matrices Γ appearing in the model action (2). (We can show all the symmetry analysis also holds for this choice of Γ's.) We look at perturbations in κ for the matrix-valued two-point function $G_-(x)$ with elements

$$G_-^{m_1 m_2}(x) = \langle \phi_-^{m_1}(u)\bar{\phi}_-^{m_2}(v)\rangle\, \chi_{u^0 \leq v^0} - \langle \bar{\phi}_-^{m_2}(u)\phi_-^{m_1}(v)\rangle\, \chi_{u^0 > v^0}\ ,$$

where $m_1, m_2 = \pm 3/2, \pm 1/2$ denote the four spin states for a baryon. The point is to see, after analyzing the zeroes of the convolution inverse two-point function $\Gamma_-(x)$, whether there are particles in the spectrum with different masses, determined by $\vec{p} = \vec{0}$ value of $\det \tilde{\Gamma}_-(p^0, \vec{p}) = 0$, for imaginary p^0.

Up to now, our analysis indicates a mass splitting in order κ^6, originating from purely spatial 'L'-type contributions e.g. with three quark lines (i.e. a baryon line) connecting the origin $(0,0,0)$ to the point $(1,0,1)$, and passing by the point $e^1 = (1,0,0)$ or $e^3 = (0,0,1)$, like the two contributions represented in Figure 1 below. If the 'L' involves the expectation

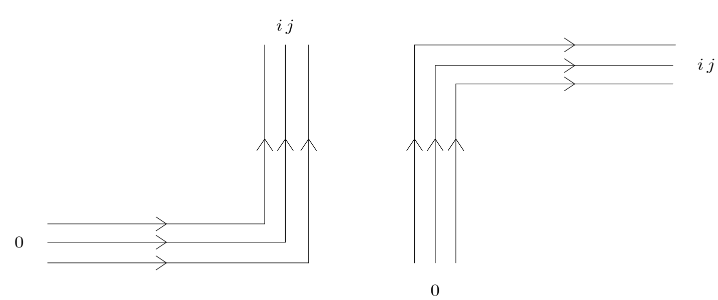

Figure 1. 'L'-type contributions involving directions i and j, $j \neq i$. Each line is oriented and all three lines have the same orientation, in each case. For $G_-^{m_1 m_2}(x \equiv ij)$, the orientation flows from the point 0 to ij, and the opposite for $G_+^{m_1 m_2}(x \equiv ij)$.

of e.g. a baryon field at the origin 0 and a field at the point ij, where ij means translation of 0 by $\pm e_i$, followed by a translation by $\pm e_j$, with $ij = (1,3)$ or $(2,3)$, or with the opposite order, we can have spin flips for the quarks in the baryons, afforded by our current Γ matrices, which originate mass splitting between the $\phi_-^{\pm 1/2}$ and $\phi_-^{\pm 3/2}$ particles. By the charge conjugation symmetry, the same holds for their anti-particles.

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