A Meson-Meson Bound State in a $2+1$ Lattice QCD Model With Two Flavors and Strong Coupling

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We consider the existence of bound states of two mesons in an imaginary-time formulation of lattice QCD. We analyze an $SU(3)$ theory with two flavors in $2+1$ dimensions and two-dimensional spin matrices. For small hopping parameter and sufficiently large glueball mass, as a preliminary, we show the existence of isoscalar and isovector meson-like particles that have isolated dispersion curves (upper gap up to near the two-particle threshold $\sim -4\ln\kappa$). The corresponding meson masses are equal up to and including $\mathcal{O}(\kappa^3)$ and are asymptotically of order $-2\ln\kappa-\kappa^2$. Considering the zero total isospin sector, we show that there is a meson-meson bound state solution to the Bethe-Salpeter equation in a ladder approximation, below the two-meson threshold, and with binding energy of order $\bar{b}\kappa^2 \simeq 0.02359\kappa^2$. In the context of the strong coupling expansion in κ, we show that there are two sources of meson-meson attraction. One comes from a quark-antiquark exchange. This is not a meson exchange, as the spin indices are not those of the meson particle, and we refer to this as a quasi-meson exchange. The other arises from gauge field correlations of four overlapping bonds, two positively oriented and two of opposite orientation. Although the exchange part gives rise to a space range-one attractive potential, the main mechanism for the formation of the bound state comes from the gauge contribution. In our lattice Bethe-Salpeter equation approach, this mechanism is manifested by an attractive distance-zero energy-dependent potential. We recall no bound state appeared in the one-flavor case, where the repulsive effect of Pauli exclusion is stronger.

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1. INTRODUCTION

To show the existence of quantum chromodynamics (QCD) as well as to determine the particle spectrum and the scattering content of the model are among the most fundamental problems in physics. Once we prove the existence of baryons and mesons, we ought to be able to understand nuclear forces from first principles. The understanding of the effective Yukawa model, given in terms of effective baryon and meson fields and single and multiple boson exchange, starting from the elementary fields for the quarks and gluons, is the key point to bridge the long standing gap between QCD and the Yukawa interaction (see Refs. [1, 2]). In the lattice approximation to QCD much progress has been made towards understanding the low-lying energy-momentum (e-m) spectrum both at the theoretical and numerical levels (see Refs. [3–11] and Refs. [12–14] dealing with numerical simulations). Particularly, in the context of $SU(3)$ lattice QCD in the Euclidean imaginary-time formulation (see Refs. [5, 6]), for small hopping parameter $0<\kappa<<1$ and sufficiently large glueball mass, in a recent series of papers given in Refs. [15–19], we started considering this problem. Believing, as it happens for confinement, that the bound state formation could be detected in this strong coupling regime and in the simplest algebraic version of the model, it was natural to consider first the one flavor case. In Ref. [15], we proved the existence of (anti-)baryons in $2+1$ dimensions and two-dimensional Pauli spin matrices by showing there is an associated isolated dispersion curve in the e-m spectrum for the model, ensuring the upper gap property, in the full baryonic sector, up to near the meson-baryon threshold $\sim -5\ln\kappa$. The corresponding (anti-)baryon asymptotic mass is $\sim -3\ln\kappa$. Analogous to the Källen-Lehman representation in continuous quantum field theory, a spectral representation was also obtained in Ref. [15] for the two-point correlation function. This is the tool that does allow us to relate the singularities of its Fourier transform, in the complex plane, to the e-m spectrum, via a Feynman-Kac formula. Furthermore, the symmetries of the model were established and analyzed. In Ref. [16], this one-particle result was extended to $d+1$ dimensions, $d=2,3$, and 4×4 Dirac spin matrices. The mass splitting among the baryon spin states was also analyzed. It was shown that the mass splitting is $18\kappa^6$, for $d=2$ and, if any, is at least of $\mathcal{O}(\kappa^7)$, for $d=3$. The meson sector was considered in Ref. [17]. The existence of mesons is manifested by an isolated dispersion curve, giving the asymptotic mass $\sim -2\ln\kappa-\kappa^2+\mathcal{O}(\kappa^4)$, and an upper gap property up to near the two-meson threshold $\sim -4\ln\kappa$. A mass splitting of order κ^4 for $d=2,3$ was also determined.

We remark that an upper gap property has not been established in the Hamiltonian formulation treatments,

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and we stress that the absence of a spectral representation makes it an extremely delicate problem to establish a (tiny) mass splitting by an analysis of the exponential decay rate for the truncated two-point function. So, among other rigorous results, these are the first important contributions emerging from these works. The above series of papers must also be seen as a necessary step to further our knowledge and consider the two-particle spectrum and to analyze the existence of two-hadron bound states, such as the deuteron.

The analysis of baryon-baryon, meson-meson and meson-baryon bound states was initiated in Refs. [18–20], also for the one-flavor case, $d = 2$ and 2×2 Pauli spin matrices. No bound state was found up to the two-particle threshold. Our bound state analysis is based on a lattice version of the Bethe-Salpeter (B-S) equation (for quantum fields in the continuum, a general discussion on this subject is found e.g. in Ref. [21], especially in chapters 13 and 14, following the main original ideas developed in Ref. [22]), involving a partially truncated four-point function, and adapting the methods developed to treat other lattice classical spin and lattice quantum field systems (see Refs. [23, 24], and Ref. [25] and references therein). The analysis for the baryon sector in the two-flavor (isospin) case, $d = 2$ and 2×2 Pauli spin matrices is done in [26], where a baryon-baryon bound state is found, for the total isospin $I = 0$ sector. In the context of the B-S equation, the attraction between the two baryons arises from gauge correlations of six overlapping bonds and from a quark-antiquark exchange. The effect of these gauge correlations is cancelled by the repulsive Pauli exclusion at a space-range zero contribution. The effective mechanism for binding comes from the quark-antiquark exchange, which gives rise to a space range-one attractive potential. It must be emphasized, however, that this is not a meson exchange, as the spin indices do not agree with those of meson particles [see the beginning of Section 3 for the meson fields]. We refer to this interaction as a quasi-meson exchange.

It is also important to remark that, although our bound state analysis was performed only at the leading order in the hopping parameter $\kappa > 0$, which we use to define what we call a ladder approximation, our results incorporate the main analytical properties and ingredients that are necessary to rigorously justify mathematically the ladder approximation as the leading approximation and to control non-perturbatively the complete model (see again Refs. [23, 24]). For instance, the B-S equation is defined in such a way that we can derive e.g. an appropriate exponential decay for the B-S kernel, with temporal rate larger than the one associated with two-particle threshold.

In this article, we continue the search for meson-meson bound states. As in [26], we use the imaginary-time formulation of lattice QCD and consider the case of $d = 2$ space dimensions, 2×2 Pauli spin matrices and *two quark flavors*. The presence of two flavors enlarges the configuration space for the elementary quark fields and weakens the effect of Pauli repulsion, making it easier for particles to bind at short-distance scales. Working in the same domain of parameters, and employing the same method as before, first, we show that there are two one-particle states, one scalar with total isospin $I = 0$ and one vector state with total isospin $I = 1$, with isolates dispersion curves $w_I(\vec{p}) = -2\ln\kappa + r_I(\kappa,\vec{p})$ $= -2\ln\kappa + \ln[1 - \frac{\kappa^2}{2}(\cos p^1 + \cos p^2)] + \mathcal{O}(\kappa^4)$ with, for $I = 0, 1$, $r_I(\kappa,\vec{p})$ real analytic in κ and each component p^j $(j = 1, 2)$. Clearly, $w_I(\vec{p}) \approx m_I + \frac{\kappa^2}{4}|\vec{p}|^2$, $|\vec{p}| \ll 1$, where $m_I = w_I(\vec{0}) = -2\ln\kappa - \kappa^2 + \mathcal{O}(\kappa^4)$. Concerning the two-particle states, we consider only the total isospin $I = 0$ sector. It is associated with less alignment, so we expect the repulsive effect of Pauli exclusion to be weaker and more favorable to binding. Using a ladder approximation to the B-S equation, here we also show that a very weakly bound meson-meson bound state occurs in the e-m spectrum, with approximate mass $-4\ln\kappa$ and binding energy of order $\bar{b}\kappa^2 \simeq 0.02359\,\kappa^2$.

Before going into detail, we give a qualitative, intuitive picture for the mechanism responsible for the meson-meson bound state formation, in the context of the strong coupling expansion in the hopping parameter (see [26, 27]).

As given below, in Section 2, the only nonlocal term in our model action is a hopping term which connects nearest neighbor lattice points x and y of the type

$$\kappa\,\bar{\psi}_A(x)[\Gamma U(g)]_{AB}\psi_B(y)\,,$$

where $\bar{\psi}$ and ψ are fermionic quark fields, A, B are sets of indices for color, spin and isospin, Γ includes a Pauli spin matrix and $U(g)$ is an SU(3) gauge matrix associated with oriented lattice bonds. There are two possible orientations for the x, y bond, one associated with a matrix $U(g) \in$ SU(3) and the other with $U(g^{-1}) = U^\dagger(g)$. Composite meson fields are defined using a quark field ψ and an anti-quark field $\bar{\psi}$, and the Fermi fields and the gauge field g are integrated out with appropriate measures, when evaluating field expectations or correlation functions. Integrals over the gauge fields vanish unless there is a multiple of three gauge matrices and, by the use of a cofactor formula, a matrix $U(g^{-1})$ behaves as a product of two $U(g)$ matrices (see Ref. [28]).

First, let us consider the meson-meson two-point correlation function. When considering an expansion for small hopping parameter κ, i.e. a strong coupling expansion, there are contributions to the meson-meson correlation that arise from chains of lattice bonds connecting the two points. For the lowest order in κ, at each lattice bond, there are two terms of the action with two gauge variables with opposite orientations, associated with a pair with a quark and an anti-quark. The appearance of quark-anti-quark pairs is a manifestation of confinement.

We now turn to the meson-meson interaction, which is detected through the temporal decay rate of an appropriate four-point meson function, with two mesons at time zero and two at time $t \neq 0$. In the strong coupling expansion for this correlation function, there are contributions coming from two sets of one-particle non-intersecting chains as described before, as well as single chains with each link having overlapping four gauge variables, two of opposite orientation. The associated gauge integral is different than that occurring in the non-intersecting chain case. There are also contributions coming from one quasi-meson exchange. To take into account all contributions, leading to precise decay rates, is difficult to control using the strong coupling expansion. To deal with this problem, we use the B-S equation, and next we discuss how the binding mechanism is manifested in our B-S equation approach to the bound state problem.

The B-S equation displays a separation between the connected and disconnected contributions and incorporates a resummation of the strong coupling expansion for each of these parts. The disconnected part is a product of exact two-point functions. The time Fourier transform of the B-S equation is, roughly speaking, a two-particle lattice Schrödinger resolvent operator equation (see Ref. [25]) with, in general, an energy dependent nonlocal potential described by minus the B-S kernel which also has an isospin dependence. Our result on the occurrence of a meson-meson bound state clearly stems from an attractive energy-dependent space range-zero potential arising from the gauge field correlations and an attractive space range-one potential arising from the quasi-meson exchange. In fact, if we erroneously take uncorrelated gauge fields a repulsive zero-range potential arises and even though the quasi-meson exchange is present, it is not strong enough to bind the two mesons. We point out that if we take the zero-range potential or the quasi-meson potential separately, each one still gives rise to a bound state, but with smaller binding energies. The presence of two flavors is very important since it weakens the effect of Pauli repulsion, and we recall that in the one-flavor case no bound state appeared. Whether or not a bound state occurs in the model in the region of parameters near or at the scaling limit (continuum QCD) and what is the mechanism for attraction are, of course, important questions still to be answered.

Before we close this section, we would like to emphasize some important features of one of our methods, that of hyperplane decoupling, which first appeared for the continuum in Ref. [22] and was adapted to the lattice in Refs. [16, 24]. In the case of simpler boson or Fermi models, it is usually clear a priori the field to be used to create excitations and/or particles. For the case of gauge and gauge-matter models, the situation is not so simple. The hyperplane decoupling method has the nice feature that it reveals the excitations and their associated fields and correlations functions; it can be used to determine which excitations are particles (isolated dispersion curves) and give their multiplicities; it can also be used in conjunction with subtraction techniques (see e.g. Ref. [16]) to show that the excitations revealed in the above steps are indeed all the excitations in the model spectrum below certain thresholds.

This paper is organized as follows. In Section 2, we introduce the model and some notation. In Section 3, we extend the one-particle results of Refs. [17, 19] to the two-flavor model and introduce some basic quantities and notation. The two-meson bound state analysis is left for Section 4. In Section 5, we discuss the isospin symmetry for correlation functions, and conclude with some remarks in Section 6. In order to make the text more readable, two important points of our methods and sample computations are presented in two appendices. In Appendix A, in the context of the hyperplane decoupling method, we show a 'product structure' property for derivatives of a special four meson correlation function, and show how this implies a good decay of the B-S kernel. Finally, we show how to obtain the ladder approximation to the B-S kernel in Appendix B. The ladder approximation corresponds to a potential presenting attractive and repulsive parts. A detailed analysis of how the balance between these parts occur under several conditions is also made in Appendix B that allows us to conclude that the main mechanism to produce attraction is very dependent on special gauge field correlation effects that enter in solving the B-S equation to this order.

2. THE MODEL

Our SU(3) QCD lattice model is the two-flavor extension of the one in Refs. [15, 16], in $d = 2$ space dimensions, with 2×2 Dirac spin matrices and one flavor. The partition function is given formally by

$$Z = \int e^{-S(\psi,\bar\psi,g)}\, d\psi\, d\bar\psi\, d\mu(g)\,,$$

where the model action $S \equiv S(\psi,\bar\psi,g)$ is

$$\begin{aligned} S \;=\; & \frac{\kappa}{2}\sum \bar\psi_{\alpha,a,f}(u)\Gamma^{\sigma e^\mu}_{\alpha\beta}\,(g_{u,u+\sigma e^\mu})_{ab}\psi_{\beta,b,f}(u+\sigma e^\mu) \\ & + \sum_{u\in\mathbb{Z}^3_o} \bar\psi_{\alpha,a,f}(u) M_{\alpha\beta}\psi_{\beta,a,f}(u) - \frac{1}{g_0^2}\sum_p \chi(g_p), \end{aligned}$$

where, besides the sum over repeated indices α, β, a and f, the first sum runs over $u\in\mathbb{Z}^3_o$, $\sigma=\pm1$ and $\mu=0,1,2$. For $F(\bar\psi,\psi,g)$, the normalized expectations are denoted by $\langle F\rangle$.

We use the same notation and convention as appears in Refs. [15, 16]. Here, we only recall that the Fermi quark fields $\psi_{\alpha,a,f}(u)$ and $\bar\psi_{\alpha,a,f}(u)$ belong to a Grassmann algebra and are specified by $\alpha = 1,2 \equiv +,-$ (*spin* index), $a = 1,2,3$ (color index) and $f = 1,2 \equiv +,-$ (flavor or isospin index). For the treatment of symmetries such as gauge, charge conjugation, parity, time-reversal and rotational symmetry, we refer to Refs. [16, 17]. Specifically, although we will keep calling ψ a quark field and $\bar\psi$ an anti-quark field, we point out that under charge conjugation $\mathcal{C}$ (an order reversing transformation!), we have $\mathcal{C}\psi_{\pm,a,f}(u) \mapsto \mp i\bar\psi_{\mp,a,f}(u)$ and $\mathcal{C}\bar\psi_{\pm,a,f}(u) \mapsto \pm i\psi_{\mp,a,f}(u)$. In addition, our model here presents global

U(2) flavor or isospin symmetry, and the transformations ($U \in$ U(2) and for † denoting the adjoint element) $\psi_{\alpha,a,f_1} \mapsto U_{f_1 f_2}\psi_{\alpha,a,f_2}$ and $\bar\psi_{\alpha,a,f_1} \mapsto \bar\psi_{\alpha,a,f_2}U^\dagger_{f_2 f_1}$ leave the action and the partition function invariant. The lattice points $u = (u^0,\vec u) = (u^0,u^1,u^2)$, are defined on the lattice with half-integer time coordinates $u \in \mathbb{Z}^3_o \equiv \mathbb{Z}_{1/2}\times\mathbb{Z}^2$, where $\mathbb{Z}_{1/2} = \{\pm 1/2, \pm 3/2, \ldots\}$. Letting e^μ, $\mu = 0,1,2$, denote the unit lattice vectors, there is a gauge group matrix $U(g_{u+e^\mu,u}) = U(g_{u,u+e^\mu})^{-1}$ associated with the directed bond $u, u+e^\mu$, and we drop U from the notation. We assume the hopping parameter $\kappa > 0$ to be small and much larger than the plaquette coupling parameter $g_0^{-2} > 0$. The parameter $m > 0$ is fixed such that $M_{\alpha\beta} = M\delta_{\alpha\beta}$ (δ denoting the Kronecker delta) and $M \equiv M(\kappa) = m + 2\kappa = 1$; $\Gamma^{\pm e^\mu} = -1 \pm \sigma^\mu$, $\mu = 0,1,2$, where the σ's are the usual Pauli matrices, and we take the diagonal Pauli matrix as σ^0. By polymer expansion methods (see Refs. [6, 29]), the thermodynamic limit of correlations exists and truncated correlations have exponential tree decay. The limiting correlation functions are lattice translational invariant. Furthermore, the correlation functions extend to analytic functions in the coupling parameters.

The quantum mechanical Hilbert space $\mathcal{H}$ and the e-m operators, starting from gauge invariant correlation functions, with support restricted to $u^0 = 1/2$, are obtained by a standard construction (see Ref. [6]). Letting $T_0^{x^0}$, $T_i^{x^i}$, $i = 1,2$, denote translation of the functions of Grassmann and gauge variables by $x^0 \geq 0$, $x \in \mathbb{Z}^3$; and for F and G only depending on coordinates with $u^0 = 1/2$, we have the Feynman-Kac (F-K) formula

$$(G, \check T_0^{x^0}\check T_1^{x^1}\check T_2^{x^2} F)_{\mathcal{H}} = \langle [T_0^{x^0}T_1^{x^1}T_2^{x^2} F]\Theta G\rangle \ ,$$

where Θ is an anti-linear operator which involves time reflection. We do not make any distinction between Grassmann, gauge variables and their associated Hilbert space vectors in our notation. As linear operators in $\mathcal{H}$, $\check T_\mu$, $\mu = 0,1,2$, are mutually commuting; $\check T_0$ is self-adjoint, with $-1 \leq \check T_0 \leq 1$, and $\check T_{j=1,2}$ are unitary, so that we write $\check T_j = e^{iP^j}$ and $\vec P = (P^1,P^2)$ is the self-adjoint momentum operator, with spectral points $\vec p \in \mathbb{T}^2 \equiv (-\pi,\pi]^2$. Since $\check T_0^2 \geq 0$, we define the energy operator $H \geq 0$ by $\check T_0^2 = e^{-2H}$. We refer to each point in the e-m spectrum associated to zero-momentum as mass. For the construction of the physical Hilbert space and a nonnegative self-adjoint transfer matrix in a finite space lattice model with a not too large hopping parameter, we refer to Ref. [30]. For the absence of a nonzero kernel of the transfer matrix, for the pure gauge model, we refer to Ref. [31].

We restrict our attention to the even subspace $\mathcal{H}_e \subset \mathcal{H}$ generated by an even number of $\hat\psi = \bar\psi$ or ψ. For the pure gauge case and small g_0^{-2}, the low-lying glueball spectrum is found in Ref. [32]. For large g_0, the glueball mass is $\approx 8\ln g_0$.

To determine the meson bound state spectrum, we first give spectral results for the meson particles in the next section.

3. THE ONE-PARTICLE ANALYSIS

We introduce the composite gauge invariant meson fields Π_ℓ, $\ell = 1,2,3,4$ (see Ref. [17] for comparison with the one-flavor case), by

$$\Pi_\ell(u) \;=\; \begin{cases} \frac{1}{\sqrt6}\,\big[\bar\psi_{-,a,+}(u)\psi_{+,a,+}(u) \\ \qquad +\bar\psi_{-,a,-}(u)\psi_{+,a,-}(u)\big] & ;\ \ell = 1 \\ \frac{1}{\sqrt3}\,\bar\psi_{-,a,+}(u)\psi_{+,a,-}(u) & ;\ \ell = 2 \\ \frac{1}{\sqrt3}\,\bar\psi_{-,a,-}(u)\psi_{+,a,+}(u) & ;\ \ell = 3 \\ \frac{1}{\sqrt6}\,\big[\bar\psi_{-,a,+}(u)\psi_{+,a,+}(u) \\ \qquad -\bar\psi_{-,a,-}(u)\psi_{+,a,-}(u)\big] & ;\ \ell = 4\,, \end{cases}$$

and the auxiliary fields μ_ℓ

$$\mu_\ell(u) \;=\; \begin{cases} \frac{1}{\sqrt6}\,\big[\psi_{-,a,+}(u)\bar\psi_{+,a,+}(u) \\ \qquad +\psi_{-,a,-}(u)\bar\psi_{+,a,-}(u)\big] & ;\ \ell = 1 \\ \frac{1}{\sqrt3}\,\psi_{-,a,+}(u)\bar\psi_{+,a,-}(u) & ;\ \ell = 2 \\ \frac{1}{\sqrt3}\,\psi_{-,a,-}(u)\bar\psi_{+,a,+}(u) & ;\ \ell = 3 \\ \frac{1}{\sqrt6}\,\big[\psi_{-,a,+}(u)\bar\psi_{+,a,+}(u) \\ \qquad -\psi_{-,a,-}(u)\bar\psi_{+,a,-}(u)\big] & ;\ \ell = 4\,. \end{cases}$$

By the model symmetry under parity or charge conjugation, we have

$$\langle \Pi_\ell(u)\rangle = 0 \ ,\ \langle \mu_\ell(u)\rangle = 0 \quad ; \quad \ell = 1,2,3,4\,. \qquad (1)$$

We note that, in the definition of Π_ℓ and μ_ℓ, the spin indices of the quark field ψ is $+$ and it is $-$ for the anti-quark field $\bar\psi$. Also, we will see that Π_ℓ has the interpretation of creating mesons. Moreover, the action of charge conjugation $\mathcal{C}$ on the Π_ℓ fields is given by $\mathcal{C}\Pi_{1,4} = -\Pi_{1,4}$ and $\mathcal{C}\Pi_{2,3} = -\Pi_{3,2}$, and the same for the μ_ℓ fields.

The motivation for the above definitions stems from the global U(2) isospin symmetry of the action. The relation of those states to the usual isospin states, i.e. of total isospin I and of z-component I_z is made in Section 6. The composite fields $\Pi_{\ell=1,2,3,4}$ can be identified with the states $(I,I_z) = (0,0),(1,1),(1,-1),(1,0)$, respectively.

From the F-K formula, for $x^0 \neq 0$ in $\mathbb{Z}$ and letting $\check T^{\vec x} \equiv \check T_1^{x^1}\check T_2^{x^2}$, we have $(k,\ell = 1,2,3,4)$

$$(\Pi_k(1/2,\vec u_1), \check T_0^{|x^0|-1}\,\check T^{\vec x}\,\Pi_\ell(1/2,\vec u_2))_{\mathcal{H}} \equiv G_{k\ell}(x)\,, \qquad (2)$$

where, for $x = (x^0 = u_2^0 - u_1^0, \vec x) \in \mathbb{Z}^3$, we have $G_{k\ell}(x) \equiv G_{k\ell}(u_1, u_2 + \vec x)$ and we are led to define the associated two-point correlation function (χ is the characteristic function and * is complex conjugation)

$$\begin{aligned} G_{k\ell}(u_1,u_2) = \;& \chi_{u_1^0 \leq u_2^0}\langle \mu_k(u_1)\,\Pi_\ell(u_2)\rangle^T + \\ & \chi_{u_1^0 > u_2^0}[\langle \Pi_k(u_1)\,\mu_\ell(u_2)\rangle^T]^*\,, \end{aligned}$$

where we used translation invariance and with an abuse of notation to write $G_{k\ell}(u,v) = G_{k\ell}(v-u)$. Here, $\langle AB\rangle^T = \langle AB\rangle - \langle A\rangle\,\langle B\rangle$ is the truncated $\langle AB\rangle$. Note that $G_{k\ell}$ is automatically truncated by Eq. (1). Also, as it is explained in detail in Section 5, using the isospin symmetry, it is seen that $G_{k\ell}$ is diagonal in the isospin indices k and ℓ, and the normalization of the composite fields Π_ℓ and μ_k has been chosen such that, at zero hopping parameter ($\kappa = 0$, after fixing $M = M(\kappa) = 1$), we have $G_{k\ell}(u,v) = \delta_{k\ell}\,\delta(u-v)$, where we use a continuum notation for the delta in lattice space-time variables.

For $x \in \mathbb{Z}^3$, we define the Fourier transform by $\tilde{G}_{k\ell}(p) = \sum_{x\in\mathbb{Z}^3} G_{k\ell}(x)e^{-ipx}$, $p \in \mathbb{T}^3$, the three dimensional torus $(-\pi,\pi]^3$. Inserting the spectral representations for $\check{T}_0$ and $\check{T}_{i=1,2}$ in Eq. (2), setting $\vec{u}_1 = \vec{0} = \vec{u}_2$, and taking the Fourier transform in $x = (x^0, \vec{x})$, we obtain the following spectral representation for the two-point function $G_{k\ell}$

$$
\begin{aligned}
\tilde{G}_{k\ell}(p) \;=\; & \tilde{G}_{k\ell}(\vec{p}) \;+\; (2\pi)^2 \\
& \times \int_{-1}^{1}\int_{\mathbb{T}^2} f(p^0,\lambda^0)\delta(\vec{p}-\vec{\lambda})d_{\lambda^0}\alpha_{\vec{\lambda},k\ell}(\lambda^0)d\vec{\lambda}\,,
\end{aligned}
\tag{3}
$$

where $\mathcal{E}$ is the product of the spectral families for the energy and momentum component operators, $f(x,y) \equiv (e^{ix}-y)^{-1} + (e^{-ix}-y)^{-1}$, $d_{\lambda^0}\alpha_{\vec{\lambda},k\ell}(\lambda^0)\; d\vec{\lambda} = d_{\lambda^0}d_{\vec{\lambda}}(\Pi_k(1/2,\vec{0}), \mathcal{E}(\lambda^0,\vec{\lambda})\Pi_\ell(1/2,\vec{0}))_{\mathcal{H}}$, and we set $\tilde{G}_{k\ell}(\vec{p}) = \sum_{\vec{x}} e^{-i\vec{p}.\vec{x}}G_{k\ell}(x^0=0,\vec{x})$. By time reversal and parity, we see that $\tilde{G}_{\ell\ell}$ is real.

The importance of the spectral representation given in Eq. (3) is that it allows us to identify complex p singularities of $\tilde{G}_{k\ell}(p)$ with points in the e-m spectrum.

Adapting the one-particle analysis given in Refs. [15, 17], the one-particle spectrum is determined by looking for the solutions to the equation $\det\tilde{\Gamma}_{k\ell}(p) = 0$, where $\Gamma_{k\ell}(x)$ is the convolution inverse to $G_{k\ell}(x)$. This analysis is performed using Rouché's theorem for the zeros of an analytic function (see also e.g. Ref. [29] for more detail). There are two distinct particles, corresponding to total isospin $I = 1, 0$, manifested by isolated dispersion curves in the e-m spectrum. The isovector or triplet particle is associated with $\Pi_{\ell=2,3,4}$ and the isoscalar or singlet particle is associated with Π_1. Using charge conjugation $\mathcal{C}$ and the isospin symmetry, we have $\langle\mu_2(x)\Pi_2(y)\rangle = \langle\mu_3(x)\Pi_3(y)\rangle$. Using a rotation of $\pi/4$, about the x-isospin axis, and the isospin symmetry, shows that $\langle\mu_2(x)\Pi_2(y)\rangle = \langle\mu_4(x)\Pi_4(y)\rangle$. Thus, the dispersion curves for the triplet components $\Pi_\ell = 2,3,4$ coincide. Moreover, since by charge conjugation $\mathcal{C}\Pi_1 \mapsto -\Pi_1$, $\mathcal{C}\Pi_2 \mapsto -\Pi_3$, $\mathcal{C}\Pi_3 \mapsto -\Pi_2$ and $\mathcal{C}\Pi_4 \mapsto -\Pi_4$, we see that the dispersion curve for each of these particles coincides with the ones for their anti-particles. Furthermore, the dispersion curves for the triplet and the singlet coincide up to and including order κ^3 and are given by

$$
\begin{aligned}
w_I(\vec{p}) \;&=\; -2\ln\kappa + r_I(\kappa,\vec{p}) \\
&=\; -2\ln\kappa + \ln[1-\tfrac{\kappa^2}{2}(\cos p^1+\cos p^2)] + \mathcal{O}(\kappa^4),
\end{aligned}
$$

with, for $I=0,1$, $r_I(\kappa,\vec{p})$ real analytic in κ and each component p^j $(j=1,2)$. Clearly, $w_I(\vec{p}) \approx m_I + \frac{\kappa^2}{4}|\vec{p}\,|^2$, $|\vec{p}| \ll 1$, where $m_I \equiv m_I(\kappa) = w_I(\vec{0})$ is the meson mass. The two masses m_0 and m_1 are equal up to and including order κ^3 but a mass splitting is expected to appear in a higher order in κ, since there is no apparent symmetry relating the associated particles. In Refs. [16, 17], we showed how to obtain the mass splitting for the one flavor case. It is convenient, for future purposes, to write $m_I = m' + \mathcal{O}(\kappa^4)$, $I = 0,1$, $m' = -2\ln\kappa - \kappa^2$. In the same way as above, the spectral measures $d_{\lambda^0}\alpha_{\vec{\lambda},k\ell}(\lambda^0)$, which are diagonal in $k\ell$, are grouped in only two distinct cases indexed by I. Using this label and separating the one-particle contribution, the spectral measures have the decomposition $d_{\lambda^0}\alpha_{\vec{\lambda},I}(\lambda^0) = Z_I(\vec{\lambda})\delta(\lambda^0 - e^{-w_I(\vec{\lambda})})d\lambda^0 + d\nu_I(\lambda^0,\vec{\lambda})$. Here, letting $\tilde{\Gamma}_{k\ell}(p)$ denote the analytic extension of $\tilde{G}_{k\ell}(p)^{-1}$, we have $Z_I(\vec{p})^{-1} = -(2\pi)^2\; e^{w_I(\vec{p})}\frac{\partial\tilde{\Gamma}_{k\ell}}{\partial\chi}(p^0 = i\chi,\vec{p})|_{\chi=w_I(\vec{p})}$, where following the notation fixed above, $k\ell = 11$ for $I=0$ and $k\ell = 22, 33, 44$, for $I=1$, such that $Z_I(\vec{p}) = (2\pi)^{-2}e^{-w_I(\vec{p})} + \mathcal{O}(\kappa^3)$, with $Z_I(\vec{p})$ also real analytic in κ and p^j, $j=1,2$. The λ^0 support of $d\nu_I(\lambda^0,\vec{\lambda})$ is contained in $|\lambda^0| \le |\kappa|^{4-\epsilon}$ and $\int_{-1}^{1} d\nu_I(\lambda^0,\vec{\lambda}) \le \mathcal{O}(\kappa^3)$. Points in the spectrum occur as p^0 singularities of $\tilde{G}_{k\ell}(p)$, for fixed $\vec{p}$, and the meson mass points occur as singularities for $p^0 = \pm iw_I(\vec{p}=\vec{0})$. Our analysis shows that points of the form $p^0 = \pi + i\chi$, $|\chi| < -(4-\epsilon)\ln\kappa$, are regular. Notice that the above measure decompositions show the dispersion curves are isolated up to $-(4-\epsilon)\ln\kappa$ (upper gap property), making possible the particle identifications. The isolated dispersion curves in the e-m spectrum associated with the Π_ℓ fields are the *only* spectrum in $\mathcal{H}_e$, up to mass $-(4-\epsilon)\ln\kappa$. This can be shown by adapting the subtraction method of Ref. [16].

4. BOUND STATE ANALYSIS

To determine the existence of $\Pi - \Pi$ meson-meson bound states, we consider the subspace of states generated by the vectors $\Pi_{\ell_1}(1/2,\vec{x}_1)\;\Pi_{\ell_2}(1/2,\vec{x}_2)$, $\ell_i = 1,2,3,4$, $i=1,2$, and we observe that, at coincident points they are not zero. In the case of one flavor baryons, the two-particle state at coincident points vanishes because of the Pauli principle. As we will see below, it will also be important to consider Clebsch-Gordan (C-G) linear combinations of these states with definite values I and I_z. From the F-K formula, for $x^0 \neq 0$, we have $(\Pi_{\ell_1}(1/2,\vec{u}_1)\Pi_{\ell_2}(1/2,\vec{u}_2), (T_0)^{|x^0|-1}\check{T}^{\vec{x}} \times\Pi_{\ell_3}(1/2,\vec{u}_3)\Pi_{\ell_4}(1/2,\vec{u}_4))_{\mathcal{H}} = \mathcal{G}_{\ell_1\ell_2\ell_3\ell_4}(x)$, where $\mathcal{G}_{\ell_1\ell_2\ell_3\ell_4}(x) = \mathcal{G}_{\ell_1\ell_2\ell_3\ell_4}(u_1,u_2,u_3+\vec{x},u_4+\vec{x})$, with $x = (x^0 = v^0-u^0,\;\vec{x}) \in \mathbb{Z}^3$, and for $u_1^0 = u_2^0 = u^0$ and

$u_3^0 = u_4^0 = v^0$,

$$\mathcal{G}_{\ell_1\ell_2\ell_3\ell_4}(u_1,u_2,u_3,u_4) = \langle \mu_{\ell_1}(u_1)\mu_{\ell_2}(u_2)\Pi_{\ell_3}(u_3)\Pi_{\ell_4}(u_4)\rangle \chi_{u^0\leq v^0} + \langle \Pi_{\ell_1}(u_1)\Pi_{\ell_2}(u_2)\mu_{\ell_3}(u_3)\mu_{\ell_4}(u_4)\rangle^* \chi_{u^0>v^0}\,.$$

At this point, we briefly outline the method for determining bound states before going further. First, we obtain a spectral representation for $\mathcal{G}_{\ell_1\ell_2\ell_3\ell_4}(x)$, and its Fourier transform $\tilde{\mathcal{G}}_{\ell_1\ell_2\ell_3\ell_4}(k)$. It is this spectral representation that allows us to relate the complex k singularities in $\tilde{\mathcal{G}}_{\ell_1\ell_2\ell_3\ell_4}(k)$ to the e-m spectrum. Next, we define and use a lattice B-S equation in a ladder approximation (see below), to search for the singularities of the four-point function below the two-meson threshold, which is approximately $-4\ln\kappa$.

Taking the Fourier transform and inserting the spectral representations for T_0, T_1 and T_2 gives

$$\begin{aligned}\tilde{\mathcal{G}}_{\ell_1\ell_2\ell_3\ell_4}(k) = \;& \tilde{\mathcal{G}}_{\ell_1\ell_2\ell_3\ell_4}(\vec{k}) + (2\pi)^2 \int_{-1}^{1}\int_{\mathbb{T}^2} f(k^0,\lambda^0)\\ &\times \delta(\vec{k}-\vec{\lambda})\, d_{\lambda^0} d_{\vec{\lambda}} (\Pi_{\ell_1}(1/2,\vec{u}_1)\\ &\times \Pi_{\ell_2}(1/2,\vec{u}_2), \mathcal{E}(\lambda^0,\vec{\lambda})\Pi_{\ell_3}(1/2,\vec{u}_3)\\ &\times \Pi_{\ell_4}(1/2,\vec{u}_4))_{\mathcal{H}}\,.\end{aligned}$$

where $\tilde{\mathcal{G}}_{\ell_1\ell_2\ell_3\ell_4}(\vec{k}) = \sum_{\vec{x}\in\mathbb{Z}^2} e^{-i\vec{k}.\vec{x}}\mathcal{G}_{\ell_1\ell_2\ell_3\ell_4}(x^0=0,\vec{x})$. The singularities in $\tilde{\mathcal{G}}_{\ell_1\ell_2\ell_3\ell_4}(k)$, for $k=(k^0=i\chi,\vec{k}=0)$ and $e^{\pm\chi}\leq 1$, are points in the mass spectrum, i.e. the e-m spectrum at system momentum zero.

To analyze $\tilde{\mathcal{G}}_{\ell_1\ell_2\ell_3\ell_4}(k)$, we follow the method of analysis for spin models as in Ref. [23]. To use a notation closer to that of Ref. [23], we relabel the time direction coordinates in $\mathcal{G}_{\ell_1\ell_2\ell_3\ell_4}(x)$ by integer labels, with $u_i^0 - 1/2 = x_i^0$, $\vec{u}_i = \vec{x}_i$, $i=1,\ldots,4$, and write $D_{\ell_1\ell_2\ell_3\ell_4}(x_1,x_2,x_3+\vec{x},x_4+\vec{x})$, $x_1^0 = x_2^0$ and $x_3^0 = x_4^0$, $x^0 = x_3^0 - x_2^0$, where x_i and x are points on the $\mathbb{Z}^3$ lattice. Now we pass to difference coordinates and then to lattice relative coordinates

$$\xi = x_2 - x_1, \quad \eta = x_4 - x_3 \quad \text{and} \quad \tau = x_3 - x_2\,,$$

to obtain

$$\begin{aligned}&D_{\ell_1\ell_2\ell_3\ell_4}(x_1,x_2,x_3+\vec{x},x_4+\vec{x})\\ &\quad = D_{\ell_1\ell_2\ell_3\ell_4}(0,x_2-x_1,x_3-x_1+\vec{x},x_4-x_1+\vec{x})\\ &\quad \equiv D_{\ell_1\ell_2\ell_3\ell_4}(\vec{\xi},\vec{\eta},\tau+\vec{x})\,,\end{aligned}$$

and $\tilde{\mathcal{G}}_{\ell_1\ell_2\ell_3\ell_4}(k) = e^{i\vec{k}.\vec{\tau}}\hat{D}_{\ell_1\ell_2\ell_3\ell_4}(\vec{\xi},\vec{\eta},k)$, where $\hat{D}_{\ell_1\ell_2\ell_3\ell_4}(\vec{\xi},\vec{\eta},k) = \sum_{\tau\in\mathbb{Z}^3} D_{\ell_1\ell_2\ell_3\ell_4}(\vec{\xi},\vec{\eta},\tau)e^{-ik.\tau}$. Explicitly, we have

$$\begin{aligned}D_{\ell_1\ell_2\ell_3\ell_4}(x_1,x_2,x_3,x_4) = D_{\ell_2\ell_1\ell_4\ell_3}(x_2,x_1,x_4,x_3) = \;&\langle \mu_{\ell_1}(x_1^0+1/2,\vec{x}_1)\mu_{\ell_2}(x_2^0+1/2,\vec{x}_2)\\ &\times \Pi_{\ell_3}(x_3^0+1/2,\vec{x}_3)\Pi_{\ell_4}(x_4^0+1/2,\vec{x}_4)\rangle\, \chi_{x_2^0\leq x_3^0}\\ &+ \langle \Pi_{\ell_1}(x_1^0+1/2,\vec{x}_1)\Pi_{\ell_2}(x_2^0+1/2,\vec{x}_2)\\ &\times \mu_{\ell_3}(x_3^0+1/2,\vec{x}_3)\mu_{\ell_4}(x_4^0+1/2,\vec{x}_4)\rangle^*\, \chi_{x_2^0>x_3^0}\,.\end{aligned} \tag{4}$$

The point of all this is that the singularities of $\tilde{\mathcal{G}}(k)$ are the same as those of $\hat{D}_{\ell_1\ell_2\ell_3\ell_4}(\vec{\xi},\vec{\eta},k)$ and the B-S equation for the four-point function and its analysis are familiar and have been treated before in Refs. [16, 23].

The above considerations also apply for linear combinations of the two-particle states.

Due to isospin symmetry orthogonality relations (see Section 6), the two-particle subspace decomposes into sectors labelled by two-particle total isospin I and z-component isospin I_z. As the one-particle states consist of an isoscalar and an isovector particle, both occurring with multiplicity one, the two-particle sector decomposes into states with $I=0,1,2$ with multiplicities two, three, and one, respectively.

Hereafter, to simplify our analysis, we will restrict our analysis to the total isospin zero sector, which is associated with small isospin alignment, and therefore has small Pauli exclusion repulsion.

For the two-particle state obtained by coupling two isospin states, we take the trivial C-G linear combination, with $x_1^0 = 0 = x_2^0$,

$$\mathcal{T}_0^\Pi(x_1,x_2) = \Pi_{00}(x_1^0+1/2,\vec{x}_1)\Pi_{00}(x_2^0+1/2,\vec{x}_2)\,,$$

and for the coupled state of two isovectors we take the C-G linear combination, with $x_1^0 = 0 = x_2^0$,

$$\begin{aligned}\mathcal{T}_1^\Pi(x_1,x_2) = \frac{1}{\sqrt{3}}\big[&\Pi_{11}(x_1^0+1/2,\vec{x}_1)\Pi_{1-1}(x_2^0+1/2,\vec{x}_2)\\ &-\Pi_{10}(x_1^0+1/2,\vec{x}_1)\Pi_{10}(x_2^0+1/2,\vec{x}_2)\\ &+\Pi_{1-1}(x_1^0+1/2,\vec{x}_1)\Pi_{11}(x_2^0+1/2,\vec{x}_2)\big]\,.\end{aligned}$$

Since we do not know a priori which linear combination of these states is more appropriate to describe a possible bound state, we consider the matrix of inner products of states, with $x_1^0 = 0 = x_2^0 = x_3^0 = x_4^0$ and $i,j=0,1$,

$$\left(\mathcal{T}_i^\Pi(x_1,x_2),\, e^{-H|x^0|}\, e^{i\vec{P}.\vec{x}}\, \mathcal{T}_j^\Pi(x_3,x_4)\right)_{\mathcal{H}}\,.$$

We note that this is analogous to degenerate perturbation theory in ordinary quantum mechanics, and the right linear combination will emerge at the end of our analysis.

The associated matrix-valued correlation function is, with $i,j=0,1$,

$$\begin{aligned}M_{ij} \;=\;& \langle \mathcal{T}_i^\mu(x_1,x_2)\mathcal{T}_j^\Pi(x_3,x_4)\rangle\, \chi_{x_2^0\leq x_3^0}\\ &+\langle \mathcal{T}_i^\Pi(x_1,x_2)\mathcal{T}_j^\mu(x_3,x_4)\rangle^*\, \chi_{x_2^0>x_3^0}\,,\end{aligned} \tag{5}$$

where $\mathcal{T}_i^\mu$ is defined similarly to $\mathcal{T}_i^\Pi$ with μ replacing Π.

In terms of expectations of the μ_ℓ and Π_ℓ fields, M_{ij} is given by, for $x_2^0 \leq x_3^0$ [a similar expression hold for $x_2^0 > x_3^0$, according to Eq. (5)], and suppressing lattice site arguments to simplify the notation,

$$
\begin{aligned}
M_{00} &= D_{1111}\,, \\
M_{01} &= -\frac{1}{\sqrt{3}}[D_{1123}+D_{1144}+D_{1132}]\,, \\
M_{10} &= -\frac{1}{\sqrt{3}}[D_{2311}+D_{4411}+D_{3211}]\,, \\
M_{11} &= \frac{1}{3}[D_{2323}+D_{2344}+D_{2332}+D_{4432} \\
&\quad +D_{4444}+D_{4423}+D_{3223}+D_{3244}+D_{3232}]\,.
\end{aligned}
$$

We now consider a B-S equation for the four-point function M. In order to have good decay properties of the B-S kernel, we modify M by performing a vacuum subtraction. By vacuum subtraction, we mean the subtraction of the quantity $\langle \mathcal{T}_i^{\mu}(x_1,x_2)\rangle \langle \mathcal{T}_j^{\Pi}(x_3,x_4)\rangle \chi_{x_2^0\le x_3^0} + \langle \mathcal{T}_i^{\Pi}(x_1,x_2)\rangle^* \langle \mathcal{T}_j^{\mu}(x_3,x_4)\rangle^* \chi_{x_2^0> x_3^0}$. From now on, in this section, we assume that this modification has been made, and continue to use the same notation M as before. The B-S equation in operator form, and in what we call the equal time representation, is

$$M \;=\; M_0 \;+\; M_0\, K\, M\,.$$

In terms of kernels, suppressing the matrix indices, for $x_1^0=x_2^0$ and $x_3^0=x_4^0$, we have

$$
\begin{aligned}
&M(x_1,x_2,x_3,x_4) = M_0(x_1,x_2,x_3,x_4) + \\
&\textstyle\int M_0(x_1,x_2,y_1,y_2)K(y_1,y_2,y_3,y_4)M(y_3,y_4,x_3,x_4) \\
&\times \delta(y_1^0-y_2^0)\delta(y_3^0-y_4^0)\, dy_1dy_2dy_3dy_4\,.
\end{aligned}
$$

Here, M_0 is obtained from M by erroneously applying Wick's theorem to the composite fields μ_ℓ and Π_ℓ in M of Eq. (5), i.e. [see Eq. (4)],

$$
\begin{gathered}
M_{0,00} = D_{0,1111} \quad , \quad M_{0,11} = D_{0,2222}\,, \\
M_{0,01} = M_{0,10} = 0\,,
\end{gathered}
$$

where D_0 is obtained by erroneously applying Wick's theorem to the μ_ℓ and Π_ℓ composite fields in D. The matrix elements of M, M_0 and K, which equals $M_0^{-1}-M^{-1}$, are to be taken as operators acting on $\ell_2^s(\mathcal{A})$, the symmetric subspace of $\ell_2(\mathcal{A})$, where $\mathcal{A}=\{(x_1,x_2)\in\mathbb{Z}^3\times\mathbb{Z}^3 | x_1^0 = x_2^0\}$. The inverses of M and M_0 are defined by separating out the $\kappa=0$ contributions, which are invertible, and using the Neumann series.

In terms of the $(\vec{\xi},\vec{\eta},\tau)$ relative coordinates and taking the Fourier transform in τ only, the B-S equation becomes (see Ref. [23])

$$
\begin{aligned}
&\hat{M}(\vec{\xi},\vec{\eta},k) = \hat{M}_0(\vec{\xi},\vec{\eta},k) \\
&\quad + \int \hat{M}_0(\vec{\xi},\vec{\xi}',k)\hat{K}(-\vec{\xi}',-\vec{\eta}',k)\hat{M}(\vec{\eta}',\vec{\eta},k)\, d\vec{\xi}'d\vec{\eta}'\,.
\end{aligned}
\tag{6}
$$

With k fixed, $\hat{M}_{ij}(\vec{\xi},\vec{\eta},k)$, etc, is taken as a matrix operator on $\ell_2(\mathbb{Z}^2)$, for $k=(k^0,\vec{k}=\vec{0})$, on the even subspace of $\ell_2(\mathbb{Z}^2)$. Eq. (6) corresponds to a matrix one-particle lattice Schrödinger operator resolvent equation. $\hat{K}_{ij}(-\vec{\xi}',-\vec{\eta}',k)$, in general, acts as minus an energy-dependent non-local potential in the non-relativistic lattice Schrödinger operator analogy.

The key to successfully solve the B-S equation is to obtain appropriate decay properties for the kernel of K. In particular, we want a long-range temporal decay faster than the two-particle decay, here $\kappa^{4|x_3^0-x_1^0|}$. As shown in Appendix A, the modification of M given above ensures that the B-S kernel K has this improved decay, as needed. In the context of the decoupling of hyperplane method [22, 24], that we use to obtain this decay, a 'product structure' of derivatives of K with respect to the hopping parameter is a very useful property. This is also derived in Appendix A.

Next, we look for a solution to the approximate equation where K is replaced by its dominant contribution, that is commonly called a *ladder* approximation L to K. These ingredients together with the control of perturbations to the ladder approximation lead to a rigorous solution of the B-S equation and two-particle spectral results for the complete model (see Ref. [16]).

We now determine L by using an expansion for K, which we now give. Using the superscript $(n)=(0),(1),...$ to denote the coefficient of κ^n and the argument 0 to denote coincident sites $x_1=x_2=x_3=x_4$, the expansion for K is obtained by writing $M=M^{(0)}+\delta M$, $M_0=M_0^{(0)}+\delta M_0$, and using the Neumann series for the inverses, as $M^{(0)}$ and $M_0^{(0)}$ are invertible. In this way, we get the expansion

$$
\begin{aligned}
K &= \left[M_0^{(0)}+\delta M_0\right]^{-1} - \left[M^{(0)}+\delta M\right]^{-1} \\
&= \left(M_0^{(0)}\right)^{-1} - \left(M^{(0)}\right)^{-1} + \sum_{n=1}^{\infty}(-1)^n \left\{ \left[\left(M_0^{(0)}\right)^{-1}\right.\right. \\
&\quad \left.\times\; \delta M_0\,\right]^n \left(M_0^{(0)}\right)^{-1} - \left[\left(M^{(0)}\right)^{-1}\delta M\right]^n \\
&\quad \left.\times\; \left(M^{(0)}\right)^{-1} \right\}\,,
\end{aligned}
\tag{7}
$$

where δM_0 and δM admit expansions in κ. For the zeroth order, we find that

$$
\begin{aligned}
M^{(0)}(x_1,x_2,x_3,x_4) = M^{(0)}(0)\delta(x_1-x_3)\delta(x_2-x_4)\,\times \\
\delta(x_1-x_2) + M_0^{(0)}(x_1,x_2,x_3,x_4) \\
\times\,[1-\delta(x_1-x_2)\delta(x_1-x_3) \\
\times\,\delta(x_1-x_4)]\,,
\end{aligned}
$$

such as it agrees with $M_0^{(0)}$ except at coincident points. Furthermore, recalling that $G_{k\ell}$ is diagonal and normalized to 1 at $\kappa=0$ and coincident points, we find

$$
\begin{aligned}
M_{0,ij}^{(0)}(x_1,x_2,x_3,x_4) \;&=\; \delta_{ij}\,[\delta(x_1-x_3)\delta(x_2-x_4) \\
&\quad + \delta(x_1-x_4)\delta(x_2-x_3)]\,,
\end{aligned}
$$

and $M^{(0)}_{0,ii}$ acts as twice the identity on $\ell_2^s(\mathcal{A})$. Thus,

$$K^{(0)}(x_1,x_2,x_3,x_4)=\left[\left(M_0^{(0)}(0)\right)^{-1}-\left(M^{(0)}(0)\right)^{-1}\right]\times\delta(x_1-x_3)\delta(x_2-x_4)\delta(x_1-x_2).$$

The matrix $M^{(0)}(0)$ is obtained by Wick's theorem in $\tilde\psi$ fields, and we warn the reader that Wick's theorem in the composite fields does not necessarily hold. With the partial truncation, we find

$$D^{(0)}_{1111}(0)=D^{(0)}_{4444}(0)=\frac{5}{3}\,,$$
$$D^{(0)}_{1123}(0)=D^{(0)}_{1144}(0)=-\frac{1}{3}\,,$$
$$D^{(0)}_{2323}(0)=1\ \ ,\ \ D^{(0)}_{2344}(0)=\frac{1}{3}\,.$$

The other needed entries are obtained using the symmetry property of $D_{\ell_1\ell_2\ell_3\ell_4}(x_1,x_2,x_3,x_4)$ given in Eq. (4) and time reversal symmetry (see Ref. [16]), which gives us $D^{(0)}_{\ell_1\ell_2\ell_3\ell_4}(0)=D^{(0)}_{\ell_3\ell_4\ell_1\ell_2}(0)$. Hence,

$$M^{(0)}(0)=\begin{pmatrix}\frac{5}{3} & \frac{\sqrt3}{3}\\ \frac{\sqrt3}{3} & \frac{7}{3}\end{pmatrix}=\frac{8}{3}P+\frac{4}{3}P_1\,,$$

where (see Appendix B) we have given the spectral decomposition; P, P_1 are the orthogonal projections associated with the eigenvalues 8/3 and 4/3, with the normalized eigenvectors $v=\frac{1}{2}(1,\sqrt3)$ and $v_1=\frac{1}{2}(\sqrt3,-1)$, respectively, such that $[M^{(0)}(0)]^{-1}=\frac{3}{8}P+\frac{3}{4}P_1$. Recalling that $M_0^{(0)}(0)=2I=2(P+P_1)$ and $[M_0^{(0)}(0)]^{-1}=\frac{1}{2}(P+P_1)$, we have

$$\begin{aligned}(M_0^{(0)})^{-1}-\left(M^{(0)}\right)^{-1} &= K^{(0)}(x_1,x_2,x_3,x_4)\\ &= K^{(0)}(0)\delta(x_1-x_3)\delta(x_2-x_4)\\ &\quad\times\delta(x_1-x_2)\\ &= (\tfrac{P}{8}-\tfrac{P_1}{4})\delta(x_1-x_3)\\ &\quad\times\delta(x_2-x_4)\delta(x_1-x_2)\,.\end{aligned}$$

The other contributions come from a temporal distance one κ^4 term, which lead to an attractive energy-dependent local potential, and a space range-one attractive potential associated with a quasi-meson exchange. These are the leading contributions and involve lengthy computations. They are determined in Appendix B. Taken together these contributions give the ladder approximation $L(x_1,x_2,x_3,x_4)$ to $K(x_1,x_2,x_3,x_4)$,

$$\begin{aligned}L(x_1,x_2,x_3,x_4)=&[\tfrac{P}{8}-\tfrac{P_1}{4}]\,[\delta(x_1-x_2)\delta(x_1-x_3)\\ &\times\delta(x_1-x_4)-\kappa^4\,\delta(x_1-x_2)\\ &\times\delta(x_3-x_4)\delta(x_2-x_3+e^0)]\\ &+\kappa^2{\textstyle\sum_{j=1,2}\sum_{\sigma=+,-}}\,\tfrac{P_-}{6}\,\delta(x_1-x_3)\\ &\times\delta(x_2-x_4)\delta(\sigma e^j+x_2-x_3)\,,\end{aligned}\tag{8}$$

where $P_-=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ is the projection on the zero total isospin arising from the composition of two vector isospin one-particle states.

We now consider the B-S equation within this approximation. For the relative coordinate B-S equation (see Refs. [18, 19]), Fourier transformed in the τ variable only (with dual variable $k=(k^0,\vec k)$, taken at $\vec k=\vec 0$), and suppressing the k^0 dependence, we have in the ladder approximation,

$$\begin{aligned}\hat M_{ij}(\vec\xi,\vec\eta)=&\hat M_{0,ij}(\vec\xi,\vec\eta)\\ &+\int\hat M_{0,ii'}(\vec\xi,\vec\xi')\hat L_{i'j'}(\vec\xi',\vec\eta')\hat M_{j'j}(\vec\eta',\vec\eta)d\vec\xi'd\vec\eta',\end{aligned}$$

where

$$\begin{aligned}\hat L(\xi,\eta,k^0)=&[\tfrac{P}{8}-\tfrac{P_1}{4}](1-\kappa^4e^{-ik^0})\delta(\vec\xi-\vec\eta)\delta(\vec\eta)\\ &+\kappa^2{\textstyle\sum_{j=1,2}\sum_{\sigma=+,-}}\,\tfrac{P_-}{6}\,\delta(\vec\xi-\vec\eta)\delta(\vec\xi-\sigma e^j).\end{aligned}$$

To determine the two-particle singularity, with $\bar m\equiv\min(m_0,m_1)=m'+\mathcal{O}(\kappa^4)$, we set $k^0=i(2\bar m-\epsilon)\equiv i\chi$, where $\epsilon>0$ is the binding energy; and our analysis will show that a bound state occurs for ϵ of order κ^2. As $m'=-2\ln\kappa-\kappa^2$, we take

$$\begin{aligned}\hat L(\xi,\eta,k^0)\simeq&(1-e^{-\epsilon-2\kappa^2})[\tfrac{P}{8}-\tfrac{P_1}{4}]\delta(\vec\xi-\vec\eta)\delta(\vec\eta)\\ &+\kappa^2{\textstyle\sum_{j=1,2}\sum_{\sigma=+,-}}\tfrac{P_-}{6}\delta(\vec\xi-\vec\eta)\delta(\vec\xi-\sigma e^j),\end{aligned}$$

and the B-S equation becomes, with $b\equiv(1-e^{-\epsilon})/\kappa^2$,

$$\begin{aligned}\hat M(\vec\xi,\vec\eta)=&\hat M_0(\vec\xi,\vec\eta)+\tfrac14(b+2)\kappa^2\hat M_0(\vec\xi,\vec0)(\tfrac12P-P_1)\\ &\times\hat M(\vec0,\vec\eta)+{\textstyle\sum_{i=1,2}}\tfrac13\kappa^2\hat M_0(\vec\xi,e^i)P_-\hat M(e^i,\vec\eta)\,.\end{aligned}$$

Here, the sum over $\sigma=\pm$ was performed using Eq. (4) to show that $\hat M(\vec\xi,\vec\eta)$, $\hat M(\vec\xi,-\vec\eta)$ and $\hat M(-\vec\xi,\vec\eta)$ satisfy the same B-S equation.

Rescaling $(\kappa^2/2)\hat M\to\hat M$ (the same for $\hat M_0$), we obtain

$$\begin{aligned}\hat M(\vec\xi,\vec\eta)\ =&\ \hat M_0(\vec\xi,\vec\eta)+\tfrac12(b+2)\hat M_0(\vec\xi,\vec0)(\tfrac12P-P_1)\\ &\times\hat M(\vec0,\vec\eta)+{\textstyle\sum_{i=1,2}}\tfrac23\hat M_0(\vec\xi,e^i)P_-\hat M(e^i,\vec\eta)\,.\end{aligned}\tag{9}$$

Below, we find that $\hat M_0$ corresponds to the resolvent of $(-1/2)$ times the lattice Laplacian. In the correspondence between the B-S equations and the Schrödinger operator resolvent equation, in Eq. (9), we see that we are dealing with the Schrödinger operator $-\Delta/2$ minus a space range-zero energy dependent potential plus a space range-one potential. The existence of bounds states is a very delicate point since these potentials are of order one.

To proceed, we develop a spectral representation for $\hat M_0$. As given in Refs. [18, 19], and explained in detail in Ref. [23], starting from the representation of the B-S equation in terms of the relative coordinates ξ, η and τ, taking the Fourier transform in the $\tau=(\tau^0,\vec\tau)$ variable, with corresponding dual variable $k=(k^0,\vec k)$, taking $\vec k=\vec 0$, and using the spectral representation for the two-point function $G_{\ell_1\ell_2}$, we obtain the following spectral representation for $\hat{D}_{0,\ell_1\ell_2\ell_3\ell_4}$

$$\begin{aligned}\hat{D}_{0,\ell_1\ell_2\ell_3\ell_4}(\vec{\xi},\vec{\eta},k^0)=&(2\pi)^2\int_{-1}^{1}\int_{-1}^{1}\int_{\mathbb{T}^2} f(k^0,\lambda^0\lambda'^0)\\ &\times\Big\{\Big[\cos\vec{p}.\vec{\xi}\cos\vec{p}.\vec{\eta}+\sin\vec{p}.\vec{\xi}\sin\vec{p}.\vec{\eta}\Big]\\ &\times\delta_{\ell_1\ell_3}\delta_{\ell_2\ell_4}\,d_{\lambda^0}\alpha_{\vec{p},\ell_1\ell_3}(\lambda^0)d_{\lambda'^0}\alpha_{\vec{p},\ell_2\ell_4}(\lambda'^0)\\ &+\Big[\cos\vec{p}.\vec{\xi}\cos\vec{p}.\vec{\eta}-\sin\vec{p}.\vec{\xi}\sin\vec{p}.\vec{\eta}\Big]\\ &\times\delta_{\ell_1\ell_4}\delta_{\ell_2\ell_3}\,d_{\lambda^0}\alpha_{\vec{p},\ell_1\ell_4}(\lambda^0)d_{\lambda'^0}\alpha_{\vec{p},\ell_2\ell_3}(\lambda'^0)\Big\}\,d\vec{p}\\ &+(2\pi)^{-2}\int_{\mathbb{T}^2}\Big\{\tilde{G}_{\ell_1\ell_3}(\vec{p})G_{\ell_2\ell_4}(\vec{p})\Big[\cos\vec{p}.\vec{\xi}\\ &\times\cos\vec{p}.\vec{\eta}+\sin\vec{p}.\vec{\xi}\sin\vec{p}.\vec{\eta}\Big]+\tilde{G}_{\ell_1\ell_4}(\vec{p})G_{\ell_2\ell_3}(\vec{p})\\ &\times\Big[\cos\vec{p}.\vec{\xi}\cos\vec{p}.\vec{\eta}-\sin\vec{p}.\vec{\xi}\sin\vec{p}.\vec{\eta}\Big]\Big\}\,d\vec{p},\end{aligned}$$

for $f(x,y)$ defined below Eq. (3).

We approximate $\hat{D}_{0,\ell_1\ell_2\ell_3\ell_4}$ by using the measure decomposition for $d_{\lambda^0}\alpha_{\vec{\lambda},k\ell}$ given in Section 3, separating out and keeping only the product of one-particle contributions, and recalling that $Z_I(\vec{p})=(2\pi)^{-2}e^{-w_I(\vec{p})}$ we have $d_{\lambda^0}\alpha_{\vec{p},k\ell}(\lambda^0)\approx Z_{k\ell}(\vec{p})\delta(\lambda^0-e^{-w_{k\ell}(\vec{p})})d\lambda^0$. Here, $k\ell=11$ corresponds to $I=0$ and $k\ell=22,33,44$ corresponds to $I=1$. We also use $\tilde{G}_{\ell_1\ell_2}(\vec{p})=\delta_{\ell_1\ell_2}+\mathcal{O}(\kappa^2)$ and the fact that the dispersion relation $w_I(\vec{p})\approx-2\ln\kappa-\kappa^2+\frac{\kappa^2}{2}(2-\cos p^1-\cos p^2)+\mathcal{O}(\kappa^4)$. In this way, the rescaled $\hat{M}_0(\vec{\xi},\vec{\eta})$ is given by

$$\hat{M}_0(\vec{\xi},\vec{\eta})=\begin{pmatrix}\hat{D}_{0,1111}(\vec{\xi},\vec{\eta}) & 0\\ 0 & \hat{D}_{0,2222}(\vec{\xi},\vec{\eta})\end{pmatrix}$$

where

$$\begin{aligned}\hat{D}_{0,1111}(\vec{\xi},\vec{\eta})&\simeq\hat{D}_{0,2222}(\vec{\xi},\vec{\eta})\\ &\simeq(2\pi)^{-2}\int_{\mathbb{T}^2}\frac{\cos\vec{p}.\vec{\xi}\cos\vec{p}.\vec{\eta}}{B}\,d\vec{p}\equiv R_0(\vec{\xi},\vec{\eta}),\end{aligned}\tag{10}$$

and we have set $B=2-\cos p^1-\cos p^2+b$. We note that in arriving at our expression for $\hat{M}_0$, we have successively approximated the $Z^2 f$ of $\hat{D}_0$ by

$$\frac{1}{(2\pi)^4}\,\frac{1}{e^{2(w_I-\bar{m})+\epsilon}-1}\simeq\frac{e^{-\epsilon}}{(2\pi)^4}\,\frac{1}{2(w_I-\bar{m})+1-e^{-\epsilon}}\,.$$

Also, in this approximation, the k^0 independent terms of $\hat{D}_0$ are dropped.

We now turn to the solution for $\hat{M}$ and the determination of bound states below $2\bar{m}$. The B-S equation is solved as in [19]. In Appendix B, we show how to go through the computation of the effective ladder potential which appears in the two last terms of the r.h.s. of Eq. (9), and analyze which is the main source of attraction for the binding mechanism to work, when a bound state solution exists for the B-S equation. Letting $m_{00}=\hat{M}_0(0,0)$, $m_{01}=\hat{M}_0(0,e^1)=\hat{M}_0(0,e^2)$ and $m_{ij}=\hat{M}_0(e^i,e^j)$, bound states occur as zeroes of the determinant of the 6×6 matrix,

$$\mathcal{W}=\begin{pmatrix} I-\frac{b+2}{2}m_{00}R & -\frac{2}{3}m_{01}P_- & -\frac{2}{3}m_{01}P_-\\ -\frac{b+2}{2}m_{01}R & I-\frac{2}{3}m_{11}P_- & -\frac{2}{3}m_{12}P_-\\ -\frac{b+2}{2}m_{01}R & -\frac{2}{3}m_{12}P_- & I-\frac{2}{3}m_{11}P_-\end{pmatrix}\tag{11}$$

where

$$R\equiv(P/2-P_1)=\frac{1}{8}\begin{pmatrix}-5 & 3\sqrt{3}\\ 3\sqrt{3} & 1\end{pmatrix}.\tag{12}$$

As for $\hat{M}_0$, letting $r_{00}=R_0(0,0)$, $r_{01}=R_0(0,e^1)=R_0(0,e^2)$, $r_{ij}=R_0(e^i,e^j)$, and setting $a_1=\frac{5}{16}(b+2)r_{00}$, $a_2=-\frac{3\sqrt{3}}{16}(b+2)r_{00}$, $a_3=-\frac{1}{16}(b+2)r_{00}$, $a_4=-\frac{2}{3}r_{01}$, $a_5=\frac{5}{16}(b+2)r_{01}$, $a_6=-\frac{3\sqrt{3}}{16}(b+2)r_{01}$, $a_7=-\frac{1}{16}(b+2)r_{01}$, $a_8=-\frac{2}{3}r_{11}$ and $a_9=-\frac{2}{3}r_{12}$; $\det\mathcal{W}$ can be written

$$\begin{vmatrix}1+a_1 & a_2 & 0 & 0 & 0 & 0\\ a_2 & 1+a_3 & 0 & a_4 & 0 & a_4\\ a_5 & a_6 & 1 & 0 & 0 & 0\\ a_6 & a_7 & 0 & 1+a_8 & 0 & a_9\\ a_5 & a_6 & 0 & 0 & 1 & 0\\ a_6 & a_7 & 0 & a_9 & 0 & 1+a_8\end{vmatrix}.$$

To calculate $\det\mathcal{W}$, we apply a Laplace expansion subsequently to the third and fifth columns. Evaluating the determinant of the resulting 4×4 matrix, the zeroes are given as solutions of

$$1+a_8-a_9 = 0\tag{13}$$

$$\left[(1+a_1)(1+a_3)-a_2^2\right](1+a_8+a_9)+2c = 0,\tag{14}$$

where $c=a_4\left[a_2a_6-(1+a_1)a_7\right]$.

The condition (13) can be written as

$$\begin{aligned}\tfrac{3}{2}&=r_{11}-r_{22}\\ &=\frac{1}{2(2\pi)^2}\int_{\mathbb{T}^2}\frac{(\cos p^1-\cos p^2)^2}{(2-\cos p^1-\cos p^2)+b}d\vec{p},\end{aligned}\tag{15}$$

and we note that there is no singularity in the integrand at $p^1=0=p^2$ for $b\geq 0$ and that the integral monotonically decreases for $b\geq 0$. A numerical integration, for $b=0$, shows that the r.h.s. of Eq. (15) is less than 3/2, so there is no bound state solution.

We now consider the condition (14). After some lengthy algebra and using the identities (see Ref. [19])

$$r_{11}+r_{12}=(b+2)r_{10}\quad,\quad r_{10}=-\frac{1}{2}+\frac{1}{2}(b+2)r_{00},$$

the condition (14) becomes $c_2r_{00}^2+c_1r_{00}+c_0=0$, where $c_2=-1-5b/4-b^2/2-b^3/16$, $c_1=-1/2-3b/4-b^2/4$ and $c_0=13/8+5b/16$. Thus, we obtain the condition $r_{00}=(-c_1\pm\sqrt{c_1^2-4c_2c_0})/(2c_2)$ and the negative root r_+ is excluded since $r_{00}\geq 0$.

To analyze the positive root r_-, we first note that

$$r_{00}=(2\pi)^{-2}\int_{\mathbb{T}^2}\frac{1}{2-\cos p^1-\cos p^2+b}\,d\vec{p}$$

is positive, monotone decreasing, analytic in $b>0$, has a logarithmic singularity at $b=0$, and asymptotically behaves as $1/(b+2)$ for large b. Putting $g(b)\equiv 12+6b+b^2$, the root

$$r_- = \frac{-2(b+1)+3\sqrt{g(b)}}{(b+2)\,(b+4)}\,, \qquad (16)$$

is $r_-(b)\simeq -1/4+3\sqrt{3}/4+\mathcal{O}(b)\simeq 1.04904+\mathcal{O}(b)$, for small b, is real analytic, monotone decreasing and asymptotically behaves as $1/b$, for large b. So, the graph of r_{00} is above the graph of r_- at $b=0$ and, for large b, using a power series expansion in $1/b$, we show that it is below. Hence, there is a bound state solution to the equation

$$(2\pi)^{-2}\int_{\mathbb{T}^2}\frac{1}{2-\cos p^1-\cos p^2+b}\,d\vec{p}-r_-=0\,. \qquad (17)$$

Graphically the solution $\bar{b}$ to Eq. (17) is seen to be unique. A numerical computation gives $\bar{b}\simeq 0.02359$, so that the bound state occurs with a binding energy given by $\epsilon=-\ln(1-\bar{b}\kappa^2)\approx\bar{b}\kappa^2=0.02359\kappa^2$, which is consistent with our approximations for the B-S equation.

Also, it is important to emphasize that, for $b>0$, the above integral admits an elliptic function representation. It is, up to a factor, the Fourier sine transform of the square of the zero order Bessel function J_0, so that we can write the r_- bound state condition as

$$r_-(b)=\frac{2}{\pi(b+2)}K\left(\frac{2}{b+2}\right)\,,$$

with $K(a)=\int_0^{\pi/2}(1-a^2\sin^2\phi)^{-1/2}\,d\phi$, $0\le a\le 1$.

5. ISOSPIN SYMMETRY AND CORRELATION FUNCTIONS

Here, we use isospin symmetry to obtain the two-point correlation function *orthogonality* relations, i.e. $\langle\mu_k(x)\Pi_\ell(y)\rangle=0$, for $k\neq\ell$, $k,\ell=1,2,3,4$. These relations are the analog of the usual Hilbert space orthogonality relations for states with different z-component of angular momentum and different total angular momentum in ordinary quantum mechanics. The μ-Π correlation functions are linear combinations of $\langle\psi_{f_1}\bar{\psi}_{f_2}\bar{\psi}_{f_3}\psi_{f_4}\rangle$, where we suppress all indices but those of isospin or flavor. We start from the identity, for arbitrary $U\in \mathrm{U}(2)$, given by

$$\langle\psi_{f_1}\bar{\psi}_{f_2}(\bar{\psi}\,U)_{f_3}(U^\dagger\,\psi)_{f_4}\rangle=\langle(U\,\psi)_{f_1}(U^\dagger\,\bar{\psi})_{f_2}\bar{\psi}_{f_3}\psi_{f_4}\rangle\,,$$

where the r.h.s. is obtained from the l.h.s. using isospin symmetry. Multiplying on the left by the tensor $w_{f_1f_2}$ and $v_{f_3f_4}$ on the right, we obtain our basic identity

$$w_{f_1f_2}\,\langle\psi_{f_1}\bar{\psi}_{f_2}\bar{\psi}_{i_3}\psi_{i_4}\rangle\,U_{i_3f_3}\,\bar{U}_{i_4f_4}\,v_{f_3f_4}= w_{i_1i_2}\,U_{i_1f_1}\,\bar{U}_{i_2f_2}\,\langle\psi_{i_1}\bar{\psi}_{i_2}\bar{\psi}_{f_3}\psi_{f_4}\rangle\,v_{f_3f_4}\,,$$

where the bar on U means complex conjugation. Denoting the $\langle\cdot\rangle_{f_1f_2;f_3f_4}$ by $T_{f_1f_2;f_3f_4}$, we can write the above as $w_{12}T_{12;34}[(U\otimes\bar{U})v]_{34}=[w(U\otimes\bar{U})]_{12}T_{12;34}v_{34}$. So, we see that the tensor product of U and $\bar{U}$ occur in distinct factors. Recall that for arbitrary $U\in\mathrm{SU}(2)$, we have (see e.g. Ref. [33])

$$\bar{U}=S\,U\,S^{-1}\ \ ,\ \ S=i\sigma_y=-S^T=\bar{S}=-S^{-1}\,,$$

so that

$$U\otimes\bar{U}=U\otimes(SUS^{-1})=(1\otimes S)(U\otimes U)(1\otimes S^{-1})\,.$$

Writing $U=e^{i\theta\sigma/2}$, where $\theta\sigma=\theta_x\sigma_x,\,\theta_y\sigma_y,\,\theta_z\sigma_z$, $\theta_{x,y,z}\in\mathbb{R}$, expanding in powers of θ and equating the linear term for $\theta\sigma=\theta_z\sigma_z$, and the sum of the θ_x^2, θ_y^2 and θ_z^2 terms, we get, with $\vec{I}=\vec{\sigma}/2=(I_x,I_y,I_z)$,

$$w_{12}T_{12;34}[(I'_z)v]_{34}=[w(I'_z)]_{12}T_{12;34}v_{34}\,, \qquad (18)$$

where $I'_z=I_{1z}\otimes 1\,+\,1\otimes(SI_{2z}S^{-1})$.

For the coefficient of the sum of quadratic terms, we have

$$w_{12}T_{12;34}[(I')^2v]_{34}=[w(I')^2]_{12}T_{12;34}v_{34}\,, \qquad (19)$$

where $(I')^2_{ij}=\sum_{k=x,y,z}[I_{ik}\otimes 1+1\otimes(SI_{jk}S^{-1})]^2$. We remark that if the tensor product representations $U\otimes U$ occurred, rather than $U\otimes\bar{U}$, then the isospin operators above are the usual I_z, the z component of total isospin, and I^2, the total isospin squared. With the usual angular momentum notation, a tensor product being understood, let us denote the usual eigenfunctions of total isospin and z-component of total isospin by the triplet

$$\chi_{10}=\frac{1}{\sqrt{2}}(\alpha\beta+\beta\alpha)\ ,\ \chi_{11}=\alpha\alpha\ ,\ \chi_{1-1}=\beta\beta\,,$$

and the singlet $\chi_{00}=\frac{1}{\sqrt{2}}(\alpha\beta-\beta\alpha)$, where $\alpha=\binom{1}{0}$ and $\beta=\binom{0}{1}$. Of course, the I^2 eigenvalue is $I(I+1)=2$ (respectively, 1) and the I_z eigenvalues are 0, 1, -1 (respectively, 0).

The corresponding eigenfunctions of $(I')^2$ are given by

$$\chi'_{Im}=(1\otimes S)\chi_{Im}\,,$$

that is

$$\chi'_{10}=\frac{1}{\sqrt{2}}(\alpha\alpha-\beta\beta)\ ,\ \chi'_{11}=\alpha(-\beta)\ ,\ \chi'_{1-1}=\beta\alpha\,,$$

and $\chi'_{00}=\frac{1}{\sqrt{2}}(\alpha\alpha+\beta\beta)$, and they have the same eigenvalues as those of the corresponding χ_{Im}'s.

Using these prime eigenfunctions in the identities (18) and (19) gives the form of the $\langle\mu_k\Pi_\ell\rangle$ correlation function as linear combinations of $\langle\psi_{f_1}\bar{\psi}_{f_2}\bar{\psi}_{f_3}\psi_{f_4}\rangle$ and also the orthogonality relations. For example, for $v=\chi'_{10}$, take $v_{f_3f_4}=\frac{1}{\sqrt{2}}[\delta_{f_3+}\delta_{f_4+}-\delta_{f_3-}\delta_{f_4-}]$, etc. Here, we use the fact that the operator $\vec{\sigma}.(S\vec{\sigma}S^{-1})$ has the same action to the left or to the right on $w_{i_1i_2}$.

We can extend our isospin analysis to two-particle states by considering the four-point function $\langle \mu_{\ell_1}\mu_{\ell_2}\Pi_{\ell_3}\Pi_{\ell_4}\rangle$. We have the identity, for $U \in \mathrm{U}(2)$,

$$\begin{aligned}&\langle \psi_{f_1}\bar\psi_{f_2}\psi_{f_3}\bar\psi_{f_4}(\bar\psi U)_{f_5}(U^\dagger\psi)_{f_6}(\bar\psi U)_{f_7}(U^\dagger\psi)_{f_8}\rangle\\ &\quad = \langle (U\psi)_{f_1}(\bar\psi U^\dagger)_{f_2}(U\psi)_{f_3}(\bar\psi U^\dagger)_{f_4}\bar\psi_{f_5}\psi_{f_6}\bar\psi_{f_7}\psi_{f_8}\rangle\,,\end{aligned}$$

or, with similar notation as before,

$$T_{1234;5678}(U\otimes\bar U\otimes U\otimes\bar U) = (U\otimes\bar U\otimes U\otimes\bar U)T_{1234;5678}\,,$$

The four-fold tensor product $U\otimes\bar U\otimes U\otimes\bar U$ can be decomposed as $(U\otimes\bar U)\otimes(U\otimes\bar U)$ and, as already seen, each $U\otimes\bar U$ factor has isospin eigenfunctions χ'_{10}, χ'_{11}, χ'_{1-1} and χ'_{00}. For the tensor product, we have a sixteen dimensional basis which can be decomposed into total isospin two, one (multiplicity three) and zero (multiplicity two). The meson-meson bound state that we have found here is a non-trivial and non-perturbative in κ linear combination of the two isospin zero states, as we have seen in Section 4.

6. FINAL REMARKS

Starting from first principles, we show the existence of isoscalar and isovector meson particles in a $2+1$ dimensional model of lattice QCD, with two-flavors, 2×2 Pauli spin matrices and in the strong coupling regime. Adapting our techniques from previous works, we can also determine the mass splitting for these meson particles.

Within a ladder approximation to the lattice Bethe-Salpeter equation, and considering the space of states corresponding to total isospin zero, we analyze the two-meson sector of the energy-momentum spectrum. We do find a meson-meson bound state below the two-particle threshold. As we showed, there are two sources of attraction. One is an energy-dependent space zero-range potential that also presents a repulsive part, and incorporates a gauge field correlation effect of the gauge integral with four overlapping gauge bonds. The other source of attraction comes from quasi-meson exchange associated with a space range-one potential.

Our analysis shows that the most important mechanism for the binding lies in the four gauge field correlation effects since, if it is erroneously replaced by the uncorrelated product of two gauge integrals each of which with only two gauge fields then there is no bound state, even though the attractive quasi-meson exchange potential is present (see Appendix B for details).

For the same model, and considering the entire two-particle total isospin space, in ref. [26], we have also shown the occurrence of a zero total isospin two-baryon bound state. We mention that the gauge correlation effects (now due to the correlation of six overlapping gauge fields) also produce an attractive zero-range potential for the two baryons. However, if theses correlation effects are neglected, the quasi-meson exchange attractive potential still dominates and produces a bound state. This is related to the fact that the coefficient of the kinetic energy is of order κ^3 for baryons as compared to κ^2 for mesons. This is all related to confinement, and we remind the reader that the free fermion kinetic energy coefficient is proportional to κ.

A natural interesting question is to see if a meson-meson bound state also occurs in the other isospin sectors $I = 1, 2$, and if a bound state of the type meson-baryon also occurs in this model. We recall that no bound state was found in the one flavor version of this model, for which the effect of the Pauli repulsion is much stronger.

Besides the fact that algebraic complexity is considerably increased in the more realistic $3+1$ dimensional case, with 4×4 Dirac spin matrices, we see no limitation for our method to work as well, and we expect to be able to prove the binding between protons and neutrons in this way. We recall that the use of a ladder as a leading approximation to the Bethe-Salpeter kernel is justified in our method, and that it also includes the main ingredients to allow us to control non-perturbatively the full model.

Although the binding of quarks to form baryons and mesons is always seen, and the one-particle spectrum is controlled in all the strong coupling versions of the model we considered, it is still a challenge to determine whether or not a $3+1$ dimensional lattice QCD model, with only two flavors, has a proton-neutron bound state, such as a deuteron, in the strong coupling regime.

APPENDIX A: DECAY OF THE B-S KERNEL

In this appendix we obtain bounds on the B-S kernel using the decoupling of hyperplane method [23].

To obtain the temporal falloff of $\mathcal{G}$ we introduce a representation with duplicate variables, which stands for replacing the truncated function $\mathcal{G}_{F,H}(x,y) = \langle F(x)H(y)\rangle - \langle F(x)\rangle\langle H(y)\rangle$ by the equivalent expression depending on a hyperplane decoupling parameter κ_p. Here, in the terms of the action connecting the hyperplane $x^0 = p$ and $x^0 = p+1$, the hopping parameter κ is replaced by the complex parameter κ_p. For $(x^0 \le p < y^0)$, we have

$$\begin{aligned}\mathcal{G}_{F,H}(x,y) &= \frac{1}{2Z^2}\int [F(x)-F'(x)]\,[H(y)-H'(y)]\\ &\quad\times\,\exp\Big\{-\textstyle\sum_{w^0=p}\kappa_p\,\big[A(\psi,\bar\psi,g,w)\\ &\quad +A(\psi',\bar\psi',g',w)\big]\Big\}\\ &\quad\times\, e^{\left[-\mathcal{S}(\psi,\bar\psi,g)-\mathcal{S}(\psi',\bar\psi',g')\right]}\,d\psi\,d\bar\psi\,d\mu(g)\\ &\quad\times\, d\psi'\,d\bar\psi'\,d\mu(g')\\ &\equiv \frac{\mathcal{N}}{2\mathcal{D}}\,,\end{aligned} \tag{A1}$$

where

$$A(\psi,\bar\psi,g,w) \;\equiv\; \tfrac{1}{2}\Big[\bar\psi_{\alpha,a,f}(w)\Gamma^{e^0}_{\alpha\beta}(g_{w,w+e^0})_{ab} \times \psi_{\beta,b,f}(w+e^0) + \bar\psi_{\alpha,a,f}(w+e^0) \times \Gamma^{-e^0}_{\alpha\beta}(g_{w+e^0,w})_{ab}\psi_{\beta,b,f}(w)\Big]\,,$$

and Z^2 is the normalization factor depending on κ_p. F' and H' are functions of the duplicate variables ψ', $\bar\psi'$ and g'. $\mathcal{S}(\psi,\bar\psi,g)$ is the action for the remaining bonds, other than the chosen one associated with the expansion parameter κ_p.

Hereafter, we will assume that we are dealing with the particular case $F(x) \equiv \mu_{\ell_1}(x)\mu_{\ell_2}(x+\vec z)$ and $H(y) \equiv \Pi_{\ell_3}(y)\Pi_{\ell_4}(y+\vec w)$. Next, we expand the numerator $\mathcal{N} \equiv \sum_{j=0}^{\infty}\mathcal{N}^{(j)}\kappa_p^j$ and denominator $\mathcal{D} \equiv \sum_{j=0}^{\infty}\mathcal{D}^{(j)}\kappa_p^j$ of Eq. (A1) in powers of κ_p to get

$$\mathcal{G}_{F,H}(x,y) = \frac{\mathcal{N}^{(2)}}{2\mathcal{D}^{(0)}}\kappa_p^2 + \left(\frac{\mathcal{N}^{(4)}}{2\mathcal{D}^{(0)}} - \frac{\mathcal{N}^{(2)}\mathcal{D}^{(2)}}{2(\mathcal{D}^{(0)})^2}\right)\kappa_p^4 + \mathcal{O}(\kappa_p^6). \tag{A2}$$

More explicitly, we have, up to $\mathcal{O}(\kappa_p^6)$ terms,

$$\begin{aligned}\mathcal{G}_{F,H}(x,y) &= \kappa_p^2 \textstyle\sum_{w^0=p}\left[\mathcal{G}_{F,\Pi_k}(x,w)\mathcal{G}_{\mu_k,H}(w+e^0,y)\right]\\ &+\kappa_p{}^4\Big\{\textstyle\sum_{\vec z\neq\vec w|w^0=p}\tfrac{1}{2}\left[\mathcal{G}_{F,\Pi_k\Pi_\ell(\vec z)}(x,w)\right.\\ &\left.\times\,\mathcal{G}_{\mu_k\mu_\ell(\vec z),H}(w+e^0,y)\right]\\ &+\textstyle\sum_{w^0=p}\tfrac{9}{16}\left[\mathcal{G}_{F,\bar h_{12}\bar h_{34}(0)}(x,w)\mathcal{G}_{h_{12}h_{34}(0),H}(w+e^0,y)\right]\\ &+\textstyle\sum_{w^0=p}\tfrac{3}{16}\left[\mathcal{G}_{F,\bar h_{12}\bar h_{34}(0)}(x,w)\mathcal{G}_{h_{14}h_{32}(0),H}(w+e^0,y)\right]\Big\}\end{aligned} \tag{A3}$$

where $h_{ij} = \psi_{-,a,f_i}\bar\psi_{+,a,f_j}/\sqrt{3}$, $\bar h_{ij} = \bar\psi_{-,a,f_i}\psi_{+,a,f_j}/\sqrt{3}$, $\mathcal{G}_{F,\Pi_k\Pi_\ell(\vec z)}(x,w) = \langle F(x)\Pi_k(w)\Pi_\ell(w+\vec z)\rangle - \langle F(x)\rangle\langle\Pi_k(w)\Pi_\ell(w+\vec z)\rangle$, and it is understood that the r.h.s. of Eq. (A3) is evaluated at $\kappa_p = 0$. We observe that expectations with a single composite field μ or Π are zero, using parity symmetry.

We now give a sample calculation for obtaining Eq. (A3). Considering the term $\mathcal{N}^{(4)}/2\mathcal{D}^{(0)}$ in Eq. (A2), we have

$$\begin{aligned}\frac{\mathcal{N}^{(4)}}{2\mathcal{D}^{(0)}} \;=\; \frac{1}{32\mathcal{D}^{(0)}}\int(F-F')(H-H')&\Bigg[\sum_{(\vec w_1,\vec w_2)|w_1\neq w_2}\\ &\times\left(A_1^+A_1^-A_2^+A_2^- + A_1^+A_1^-A'^+_2A'^-_2\right)\\ &+\tfrac{1}{2}\textstyle\sum_{\vec w_1}(A_1^+)^2(A_1^-)^2\Big]\\ &\times e^{-\mathcal{S}-\mathcal{S}'}d\psi d\bar\psi d\mu(g)d\psi' d\bar\psi' d\mu(g')\,,\end{aligned} \tag{A4}$$

where $A_j^+ \equiv \bar\psi_{\alpha,a,f}(w_j)\;\Gamma^{e^0}_{\alpha\beta}(g_{w_j,w_j+e^0})_{ab}\psi_{\beta,b,f}(w_j+e^0)$, $A_j^- \equiv \bar\psi_{\alpha,a,f}(w_j+e^0)\;\Gamma^{-e^0}_{\alpha\beta}(g_{w_j+e^0,w_j})_{ab}\psi_{\beta,b,f}(w_j)$ and F, $\mathcal{S}$ are abbreviated notations for $F(x)$, $\mathcal{S}(\psi,\bar\psi,g)$, respectively, and similarly for F', H, H' and $\mathcal{S}'$.

In the sequel, we only compute the third term in Eq. (A4). The computations are similar for the other terms, except for the gauge integrals. In terms of expectations, the third term in Eq. (A4) is given by

$$\begin{aligned}\Lambda &\equiv \tfrac{1}{64}\textstyle\sum_{\vec w_1}\langle(F-F')(H-H')(A_1^+)^2(A_1^-)^2\rangle\\ &= \tfrac{1}{64}\textstyle\sum_{\vec w_1}\langle(F-F')\,\bar\psi_{\alpha_1,a_1,f_1}\psi_{\beta_2,b_2,f_2}\bar\psi_{\alpha_3,a_3,f_3}\\ &\quad\times\psi_{\beta_4,b_4,f_4}(u)\rangle\;\langle\psi_{\beta_1,b_1f_1}\bar\psi_{\alpha_2,a_2,f_2}\psi_{\beta_3,b_3,f_3}\\ &\quad\times\bar\psi_{\alpha_4,a_4,f_4}(v)(H-H')\rangle\;\Gamma^{e^0}_{\alpha_1\beta_1}\Gamma^{-e^0}_{\alpha_2\beta_2}\Gamma^{e^0}_{\alpha_3\beta_3}\Gamma^{-e^0}_{\alpha_4\beta_4}\\ &\quad\times\textstyle\int g_{a_1b_1}g^{-1}_{a_2b_2}g_{a_3b_3}g^{-1}_{a_4b_4}d\mu(g)\,,\end{aligned} \tag{A5}$$

where we used the notation $u = (p,\vec w_1)$ and $v = (p+1,\vec w_1)$.

Now, computing the gauge integral in Eq. (A5) using

$$\begin{aligned}\mathcal{I}_4 \;&=\; \textstyle\int g_{a_1b_1}g^{-1}_{a_2b_2}g_{a_3b_3}g^{-1}_{a_4b_4}\,d\mu(g)\\ &=\; \tfrac{1}{8}\left[\delta_{a_1b_2}\delta_{a_3b_4}\delta_{b_1a_2}\,\delta_{b_3a_4} + (a_2\rightleftarrows a_4, b_2\rightleftarrows b_4)\right]\\ &\quad-\tfrac{1}{24}\left[\delta_{a_1b_2}\delta_{a_3b_4}\delta_{b_1a_4}\,\delta_{b_3a_2} + (a_2\rightleftarrows a_4, b_2\rightleftarrows b_4)\right],\end{aligned} \tag{A6}$$

and taking into account the explicit matrix structure of $\Gamma^{\sigma e^0}$ ($\sigma = \pm1$), we obtain

$$\begin{aligned}\Lambda &= \tfrac{1}{4}\textstyle\sum_{\vec w_1}\langle(F-F')\bar\psi_{-,a_1,f_1}\psi_{+,a_1,f_2}\bar\psi_{-,a_3,f_3}\psi_{+,a_3,f_4}(u)\rangle\\ &\quad\times\langle\psi_{-,b_1,f_1}\bar\psi_{+,b_1,f_2}\psi_{-,b_3,f_3}\bar\psi_{+,b_3f_4}(v)(H-H')\rangle + \tfrac{1}{12}\\ &\quad\times\textstyle\sum_{\vec w_1}\langle(F-F')\bar\psi_{-,a_1,f_1}\psi_{+,a_1,f_2}\bar\psi_{-,a_3,f_3}\psi_{+,a_3,f_4}(u)\rangle\\ &\quad\times\langle\psi_{-,b_1,f_1}\bar\psi_{+,b_1,f_4}\psi_{-,a_2,f_3}\bar\psi_{+,a_2,f_2}(v)(H-H')\rangle\,.\end{aligned} \tag{A7}$$

For the remaining terms in Eq. (A4), we need

$$\mathcal{I}_2 = \int g_{a_1b_1}g^{-1}_{a_2b_2}\,d\mu(g) = \frac{1}{3}\,\delta_{a_1b_2}\,\delta_{a_2b_1}\,. \tag{A8}$$

For the evaluation of gauge integrals of the above types, see [28].

From now on, we restrict our attention to the isospin zero sector. Upon taking the two isospin zero states for F and H, the above κ_p^2 term becomes,

$$\sum_{k,\vec w}\langle[\mathcal{T}_i^{\mu}(x_1x_2)]^T\,\Pi_k(p,\vec w)\rangle\langle\Pi_k(p+1,\vec w)\left[\mathcal{T}_j^{\Pi}(x_3x_4)\right]^T\rangle\kappa_p^2,$$

with $\left[\mathcal{T}_j^{\Pi}(x_3x_4)\right]^T = \mathcal{T}_j^{\Pi}(x_3x_4) - \langle\mathcal{T}_j^{\Pi}(x_3x_4)\rangle$. By the orthogonality relations for total isospin, only the $k=1$ term contributes, and each $\langle\ldots\rangle^{(0)}$ factor is separately zero by parity symmetry.

Now, we consider the κ_p^4 term. We note that if the $\Pi_k\Pi_\ell$ or $\mu_k\mu_\ell$ fields are replaced by new fields which are obtained from a real orthogonal transformation acting on the original fields, then the same derivative product structure holds for the new fields. In this way, we first make the identification $(f_1,f_2)\to i$ and $(f_3,f_4)\to j$, for $i,j = 1,\ldots,4$, with $(+,+)\equiv 1$, $(+,-)\equiv 2$, $(-,+)\equiv 3$ and $(-,-)\equiv 4$. Next, we define the composite fields $\bar M_i \equiv \bar\psi_{-,a,f_1}\psi_{+,a,f_2}/\sqrt{3}$, $M_i \equiv \psi_{-,a,f_1}\bar\psi_{+,a,f_2}/\sqrt{3}$, and the change of variables is represented by the 4×4 orthogonal matrix B, with entries $\sqrt{2}B_{11} = \sqrt{2}B_{14} = \sqrt{2}B_{41} = -\sqrt{2}B_{44} = B_{22} = B_{33} = 1$ and zero for the remaining elements, which acts as $\bar{M}_i \to \Pi_m B^t_{mi}$ and $M_i \to B_{in}\mu_n$. Next, we transform to the fields $\Pi_{00} = \Pi_1$, $\Pi_{11} = -\Pi_2$, $\Pi_{1-1} = \Pi_3$, $\Pi_{10} = \Pi_4$, and note that the tensor product of the transformation is real orthogonal. Finally, we go to the fields $\mathcal{T}^{\Pi}_i$, which are related to the $\Pi_{I_1m_1}\Pi_{I_2m_2}$ fields by the C-G coefficients. These C-G fields are again associated with a real orthogonal transformation. We therefore obtain, for Eq. (A7),

$$\begin{aligned}\Lambda \;=\; & \tfrac{9}{16}\textstyle\sum_{k,\vec{w}_1}\langle[\mathcal{T}^{\mu}_i]^T\left[\mathcal{T}^{\Pi}_k\right]^T(p,\vec{w}_1)\rangle\langle[\mathcal{T}^{\mu}_k]^T(p+1,\vec{w}_1)\\ & \times\left[\mathcal{T}^{\Pi}_j\right]^T\rangle + \tfrac{3}{16}\textstyle\sum_{k,\ell,\vec{w}_1}\langle[\mathcal{T}^{\mu}_i]^T\left[\mathcal{T}^{\Pi}_\ell\right]^T(p,\vec{w}_1)\rangle\\ & \times C_{\ell k}\langle[\mathcal{T}^{\mu}_k]^T(p+1,\vec{w}_1)\left[\mathcal{T}^{\Pi}_j\right]^T\rangle\,,\end{aligned}$$

with

$$C = (C_{\ell k}) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix},$$

and we also note that C can be rewritten in terms of the orthogonal projections P and P_1

$$C = -P + P_1.$$

We finally have for the fourth derivative of M the result

$$\begin{aligned} M^{(4)}&(x_1,x_2,x_3,x_4) = \tfrac{1}{2}\textstyle\sum_{\vec{w}_1,\vec{w}_2|\vec{w}_1\neq\vec{w}_2}\\ &\times M^{(0)}(x_1,x_2,(p,\vec{w}_1),(p,\vec{w}_2))\\ &\times M^{(0)}((p+1,\vec{w}_1),(p+1,\vec{w}_2),x_3,x_4)\\ &+\tfrac{9}{16}\textstyle\sum_{\vec{w}} M^{(0)}(x_1,x_2,(p,\vec{w}),(p,\vec{w}))\\ &\times M^{(0)}((p+1,\vec{w}),(p+1,\vec{w}),x_3,x_4)\\ &+\tfrac{3}{16}\textstyle\sum_{\vec{w}} M^{(0)}(x_1,x_2,(p,\vec{w}),(p,\vec{w}))\\ &\times CM^{(0)}((p+1,\vec{w}),(p+1,\vec{w}),x_3,x_4)\;.\end{aligned} \tag{A9}$$

The second and third term in the r.h.s of Eq. (A9) can be combined to give

$$\begin{aligned} M^{(4)}&(x_1,x_2,x_3,x_4) = \tfrac{1}{2}\textstyle\sum_{\vec{w}_1,\vec{w}_2|\vec{w}_1\neq\vec{w}_2}\\ &\times M^{(0)}(x_1,x_2,(p,\vec{w}_1),(p,\vec{w}_2))\\ &\times M^{(0)}((p+1,\vec{w}_1),(p+1,\vec{w}_2),x_3,x_4)\\ &+\tfrac{1}{2}\textstyle\sum_{\vec{w}} M^{(0)}(x_1,x_2,(p,\vec{w}),(p,\vec{w}))\\ &\times \Gamma M^{(0)}((p+1,\vec{w}),(p+1,\vec{w}),x_3,x_4)\;.\end{aligned} \tag{A10}$$

where Γ is a matrix acting on the isospin degree of freedom given by

$$\Gamma = \frac{3}{4}P + \frac{3}{2}P_1.$$

Now, we explain how the product structure of the r.h.s. of Eq. (A10) will lead to the improved decay for the B-S kernel K. For this, we write Eq. (A10) schematically as

$$M^{(4)} = \frac{1}{2}\,M^{(0)}\;\circledast\;M^{(0)} + \frac{1}{2}\,M^{(0)}\;\circledcirc\;\Gamma M^{(0)}\,, \tag{A11}$$

where the products $\circledast$ and $\circledcirc$ take into account the sum restrictions $\vec{w}_1 \neq \vec{w}_2$ and $\vec{w}_1 = \vec{w}_2 = \vec{w}$, respectively. The second term in Eq. (A11) comes from the contribution of four overlapping gauge bonds (two positively oriented and the other two with negative orientation) between the points $(p,\vec{w})$ and $(p+1,\vec{w})$, while the first sum comes from the two pairs of distinct contributions with two overlapping bonds of opposite orientations linking $(p,\vec{w}_1)$ to $(p+1,\vec{w}_1)$ and $(p,\vec{w}_2)$ to $(p+1,\vec{w}_2)$, respectively. These are the only two allowed gauge-invariant contributions and it is the difference in the gauge integral contribution in these two cases that gives rise to the matrix Γ that appear on the right hand side of Eq. (A11).

For $M^{(4)}_0$, we obtain the product structure

$$M^{(4)}_0 = \frac{1}{2}\,M^{(0)}_0\;\circ\;M^{(0)}_0\,, \tag{A12}$$

where $\circ$ means an unrestricted ordered sum over $\vec{w}_1$ and $\vec{w}_2$; and we note that the coefficients of the first product term of Eq. (A11) and the one in Eq. (A12) are equal.

Now, letting $\Lambda = M^{-1}$ and $\Lambda_0 = M_0^{-1}$ and remarking that $K^{n=0,1,2,3} = 0$, when calculating $K^{(4)}$ we have, for $x^0_1 = x^0_2 < x^0_3 = x^0_4$,

$$\begin{aligned} K^{(4)}&(x_1,x_2,x_3,x_4) =\\ &= \left(\Lambda^{(0)}_0 M^{(4)}_0\Lambda^{(0)}_0 - \Lambda^{(0)}M^{(4)}\Lambda^{(0)}\right)(x_1,x_2,x_3,x_4)\\ &= \tfrac{1}{2}\Big(\Lambda^{(0)}_0 M^{(0)}_0 \circledcirc M^{(0)}_0\Lambda^{(0)}_0\\ &\quad -\Lambda^{(0)}M^{(0)}\circledcirc\Gamma M^{(0)}\Lambda^{(0)}\Big)(x_1,x_2,x_3,x_4)\\ &= \tfrac{1}{2}(I-\Gamma)\textstyle\sum_{\vec{w}}\delta(x_1,(p,\vec{w}))\,\delta(x_3,(p+1,\vec{w}))\\ &\quad\times\delta(x_2,(p,\vec{w}))\,\delta(x_4,(p+1,\vec{w}))\,,\end{aligned}$$

which gives zero for temporal distance greater than one, i.e. $|x^0_1 - x^0_3| > 1$.

It turns out that the fifth derivative of K is also zero due to the imbalance of Grassmann fields $\bar{\psi}$ and ψ in the expectations.

Finally, using joint analyticity in the parameters κ_p, $x^0_1 \le p < x^0_3$, and Cauchy bounds for estimating the κ_p derivatives gives the decay $\kappa^{4+6(|x^0_3-x^0_1|-1)}$, $|x^0_1 - x^0_3| > 1$, upon setting all the κ_p equal to κ. We note that for $|x^0_1 - x^0_3| = 1$ due to the non-vanishing of $K^{(4)}(x_1,x_2,x_3,x_4)$ we get only the decay $\kappa^{4|x^0_3-x^0_1|}$. By a procedure similar to that of [23] we can improve this decay taking into account also derivatives of κ_q linking distinct points in the space direction, i.e. we replace the κ in the action S by κ_q for all bounds connecting the points $x^i = q$ and $x^i = q+1$ $(i = 1,2)$, the final result is

$$|K(x_1,x_2,x_3,x_4)| \le \begin{cases} \kappa^{4|x^0_3-x^0_1|+|\vec{x}_1+\vec{x}_2-\vec{x}_3-\vec{x}_4|_1+|\vec{x}_2-\vec{x}_1|_1+|\vec{x}_4-\vec{x}_3|_1}\,, \\ \qquad |x^0_3-x^0_1|\le 1\,; \\ \kappa^4\,\kappa^{6(|x^0_3-x^0_1|-1)+|\vec{x}_1+\vec{x}_2-\vec{x}_3-\vec{x}_4|_1+|\vec{x}_2-\vec{x}_1|_1+|\vec{x}_4-\vec{x}_3|_1}\,, \\ \qquad |x^0_3-x^0_1|>1\,, \end{cases} \tag{A13}$$

where we use the norm $|\vec{x}|_1 \equiv |x^1| + |x^2|$.

APPENDIX B: THE LADDER APPROXIMATION TO K

In this appendix we show how to obtain the ladder approximation L to K established and used in Section 4. Within this approximation, here we also make an analysis of the various terms entering in the ladder potential and their relation with the mechanism of attraction which is responsible for the formation of a bound state.

Our starting point is Eq. (A13). The decay of K is the main restriction for picking up the contributions that enter in the definition of L. This is so because, in Eq. (6), we take the Fourier transform only in the τ variable, with dual variable k. Hence, we must also consider the energy-dependent factor $e^{-i\tau^0 k^0}$ multiplying $K(\vec{\xi},\vec{\eta},\tau)$ in the expression for $\hat{K}(\vec{\xi},\vec{\eta},k^0)$. Since we are considering energies with imaginary parts close to integer multiples of the mass scale $m' \simeq -\ln\kappa^2$, we may get negative powers of κ from $e^{-i\tau^0 k^0}$, which depend on the difference $x_3^0 - x_2^0 \equiv \tau^0$. In this way, controlling the product $e^{-i\tau^0 k^0} K(\vec{\xi},\vec{\eta},\tau)$, we find the lowest order contributions to K. These are given by the following configurations:

(i) $\tau^0 = 0$, $|2\vec{\tau}+\vec{\xi}+\vec{\eta}| + |\vec{\xi}| + |\vec{\eta}| = 0 \Rightarrow |K| = \mathcal{O}(1)$ and $e^{(2m'-\epsilon)|\tau^0|} = 1$, contributing to $\hat{K}(\vec{\xi},\vec{\eta})$ with $(P/8 - P_1/4)\,\delta(\vec{\xi})\delta(\vec{\eta})\delta(\tau)$, which corresponds with a local zero range potential that has both an attractive and a repulsive part;

(ii) $\tau^0 = 0$, $|2\vec{\tau}+\vec{\xi}+\vec{\eta}| + |\vec{\xi}| + |\vec{\eta}| = 2 \Rightarrow |K| = \mathcal{O}(\kappa^2)$ and $e^{(2m'-\epsilon)|\tau^0|} = 1$, contributing to $\hat{K}(\vec{\xi},\vec{\eta})$ with $\kappa^2(P_-/6)\,\delta(\vec{\xi}-\vec{\eta})\delta(\vec{\xi}-\sigma e^i)\delta(\tau+\sigma e^i)$. This corresponds to a space range-one potential associated with a quasi-meson exchange;

(iii) $|\tau^0| = 1$, $|2\vec{\tau}+\vec{\xi}+\vec{\eta}| + |\vec{\xi}| + |\vec{\eta}| = 0 \Rightarrow |K| = \mathcal{O}(\kappa^4)$ and $e^{(2m'-\epsilon)|e^0|} = e^{-(\epsilon+2\kappa^2)}/\kappa^4$, contributing to $\hat{K}(\vec{\xi},\vec{\eta})$ with $e^{-(\epsilon+2\kappa^2)}(P_1/4 - P/8)\,\delta(\vec{\xi})\delta(\vec{\eta})\delta(\tau - e^0)$. This corresponds to an energy-dependent zero-range potential. Its calculation takes into account the effect of gauge field correlations with four overlapping bonds;

(iv) $|\tau^0| = 1$, $|2\vec{\tau}+\vec{\xi}+\vec{\eta}| + |\vec{\xi}| + |\vec{\eta}| = 2 \Rightarrow |K| = \mathcal{O}(\kappa^6)$ and $e^{(2m'-\epsilon)|e^0|} = e^{-(\epsilon+2\kappa^2)}/\kappa^4$;

(v) $|\tau^0| = 2$, $|2\vec{\tau}+\vec{\xi}+\vec{\eta}| + |\vec{\xi}| + |\vec{\eta}| = 0 \Rightarrow |K| = \mathcal{O}(\kappa^{10})$ and $e^{(2m'-\epsilon)|2e^0|} = e^{-2(\epsilon+2\kappa^2)}/\kappa^8$.

The contributions (iv) and (v) can be shown to be zero using the Neumann series expansion for M^{-1} and M_0^{-1} [see equation (7)] and recalling that $\delta M^{(2)}$ and $\delta M_0^{(2)}$ are at least $\mathcal{O}(\kappa^2)$. Hence, we need to consider in the Neumann expansion for M^{-1} (M_0^{-1}) at most products of three and five $\delta M^{(2)}$ ($\delta M_0^{(2)}$) for (iv) and (v), respectively. In the following, we use the notation $\vec{u}_n = (y_1, y_2, \ldots y_n)$ for an ordered n-uple of numbers, such that $y_i = 2m_i$ ($i = 1,\ldots,n$), $m_i \in \mathbb{N}$, subject to the constraint $\sum_{i=1}^n y_i = 6$ for (iv) and $\sum_{i=1}^n y_i = 10$ for (v). In this way, (iv) is given by the sum of the contributions $\delta M^{(6)}$, $-\sum_{\vec{u}_2}\delta M^{(y_1)}\delta M^{(y_2)}$ and $\delta M^{(2)}\delta M^{(2)}\delta M^{(2)}$ and similarly for $[M_0]^{-1}$ omitting the factors $[M^{(0)}]^{-1}$ here and in what follows below. For (iv) we give two examples that shows a non-trivial cancellation in the expansion for K. As shown above, we must fulfill the condition $|2\vec{\tau}+\vec{\xi}+\vec{\eta}| + |\vec{\xi}| + |\vec{\eta}| = 2$. In this way, we first consider $x_1 = 0$, $x_2 = e^1$, $x_3 = e^0$ and $x_4 = e^0 + e^1$. The non zero contribution $\delta M^{(6)}$ is cancelled by the next term $-[\delta M^{(2)}\delta M^{(4)} + \delta M^{(4)}\delta M^{(2)}]$ in the Neumann series and each term in the expansion for $[M_0]^{-1}$ is zero up to this order. Next, if $x_1 = 0 = x_2$, $x_3 = e^0$ and $x_4 = e^0 + e^1$, we get that $\delta M^{(6)}$ is cancelled by the corresponding expression coming from $[M_0]^{-1}$, which is $-\delta M_0^{(6)}$ and similarly for $-\sum_{\vec{u}_2}\delta M^{(y_1)}\delta M^{(y_2)}$. For (v), we get $\delta M^{(10)}$, $-\sum_{\vec{u}_2}\delta M^{(y_1)}\delta M^{(y_2)}$, $\sum_{\vec{u}_3}\delta M^{(y_1)}\delta M^{(y_2)}\delta M^{(y_3)}$, $-\sum_{\vec{u}_4}\delta M^{(y_1)}\delta M^{(y_2)}\delta M^{(y_3)}\delta M^{(y_4)}$ and, finally, $\delta M^{(2)}\delta M^{(2)}\delta M^{(2)}\delta M^{(2)}\delta M^{(2)}$. In this case, each contribution is zero and the same happens for $[M_0]^{-1}$.

The contributions of items (i)-(iii) do not vanish and do contribute to K. In the sequel, in order to give a sample computation, we consider only the second one. The first and the third can be obtained similarly. Diagrammatically, we have for item (ii) the following contributions: a) $\mu\,\mu(0) \rightleftharpoons \Pi\,\Pi(e^1)$; b) $\mu(0) \rightleftharpoons \mu\,\Pi\,\Pi(e^1)$; c) $\Pi(0) \rightleftharpoons \mu\,\mu\,\Pi(e^1)$ and d) $\mu\Pi(0) \rightleftharpoons \mu\Pi(e^1)$, with $\rightharpoonup$ (respectively, $\leftharpoondown$) indicating the term in the expansion of the action S connecting the points $x = 0$ (respectively, $x = e^1$) and $x = e^1$ (respectively, $x = 0$). The configuration a) is seen to give zero, as follows. Expanding δM in κ, and performing the gauge integrals, we obtain

$$\tfrac{\kappa^2}{12}\langle \mu\mu\bar{\psi}_{\alpha_1,a,f_1}\psi_{\beta_2,a,f_2}\rangle^{(0)} \\ \times \langle \psi_{\beta_1,b,f_1}\bar{\psi}_{\alpha_2,b,f_2}\Pi\,\Pi\rangle^{(0)}\Gamma^{e^1}_{\alpha_1\beta_1}\Gamma^{-e^1}_{\alpha_2\beta_2}\,.$$

where, again, $\langle\ldots\rangle^{(0)}$ means $\langle\ldots\rangle$ setting $\kappa = 0$ in the hopping term in the action S. In the above expression, we must have $\alpha_1 = \beta_2$ (by parity symmetry), but by imbalance of fermion fields in the spin components we obtain zero. For contributions b) and c), we have that $\left[M^{(0)}\right]^{-1}\delta M^{(2)}\left[M^{(0)}\right]^{-1}$ is cancelled out by the corresponding contribution coming from $[M_0^{(0)}]^{-1}$ in the Neumann expansion for calculating K [see Eq. (7)]. The only non-zero contribution comes from d). For this case, expanding $\langle\mu(0)\mu(e^1)\Pi(0)\Pi(e^1)\rangle$ in powers of κ and considering, for example, $\delta M_{00}^{(2)}$, the $k = 0 = \ell$ (associated with the zero total isospin subspace) entry of the matrix $\delta M_{k\ell}^{(2)}$, we have

$$\delta M_{00}^{(2)} = \tfrac{\kappa^2}{12}\langle \mu_1\Pi_1\bar{\psi}_{\alpha_1,a,f_1}\psi_{\beta_2,a,f_2}(0)\rangle^{(0)} \\ \times \langle \psi_{\beta_1,b,f_1}\bar{\psi}_{\alpha_2,b,f_2}\mu_1\Pi_1(e^1)\rangle^{(0)}\Gamma^{e^1}_{\alpha_1\beta_1}\Gamma^{-e^1}_{\alpha_2\beta_2}\,.$$

Now, using the explicit structure of the Γ matrices appearing in the hopping term, it follows that

$$\delta M_{00}^{(2)} = \frac{\kappa^2}{12} \langle \mu_1 \Pi_1 (\bar{\psi}_{+,a,f_1} \psi_{+,a,f_2} - \bar{\psi}_{-,a,f_1} \psi_{-,a,f_2})(0) \rangle^{(0)} \times \langle (\psi_{+,b,f_1} \bar{\psi}_{+,b,f_2} - \psi_{-,b,f_1} \bar{\psi}_{-,b,f_2}) \mu_1 \Pi_1 (e^1) \rangle^{(0)}.$$

Finally, using the definitions of μ_1 and Π_1 of Section 3, we obtain

$$\langle \mu_1 \Pi_1 (\bar{\psi}_{+,a,f_1} \psi_{+,a,f_2} - \bar{\psi}_{-,a,f_1} \psi_{-,a,f_2})(0) \rangle^{(0)} = 0\,.$$

Then $\delta M_{00}^{(2)} = 0$. We also get $\delta M_{01}^{(2)} = 0 = \delta M_{10}^{(2)}$ and $\delta M_{11}^{(2)} = \kappa^2/3$. In this way, after a lengthy computation, we obtain $K^{(2)}(0, e^1, 0, e^1) = (1/4)\delta M^{(2)}(0, e^1, 0, e^1) = (\kappa^2/12)P_-$. Taking into account all the contributions of the types $\delta M^{(2)}(0, \sigma e^i, 0, \sigma e^i)$ and $\delta M^{(2)}(0, \sigma e^i, \sigma e^i, 0)$ leads to the last term in Eq. (8). This term is associated with a range-one attractive potential arising from the exchange of one quasi-meson particle, and acts in the two-vector meson space, as the projector P_- occurs.

Coming back to the contributions of items (i) and (iii), which give rise to the first and second terms in Eq. (8), we observe they can be combined together using $1-e^{-\kappa^2} = \kappa^2 + \mathcal{O}(\kappa^4)$ to yield also to an order κ^2 effective potential term. Since, from Eq. (12), $(P/2 - P_1) \equiv R$ mixes the scalar and the vector total isospin subspaces, and has eigenvalues $+1/2$ and -1, this effective potential shows both attractive and repulsive parts, and is also energy dependent.

One important question is to understand how the overall attractive parts counterbalance with the repulsive one so that a bound state still shows up. For this, we have solved numerically the problem considering only the joint effect of contributions (i) and (iii), and neglecting the exchange part (ii). Here, the quantity r_- of Eq. (16) is replaced by $r' \equiv 4/(b+2)$ and, instead of Eq. (17), we obtain the condition $r_{00} = r'$ which gives the solution $b' \simeq 0.00006$. As expected, since the attraction is reduced in this case, we have $b' < \bar{b}$, and a considerably smaller binding energy.

Moreover, to see how the attractive potential associated with the contribution of the quasi-meson exchange (ii) compares with the attractive part of the zero range potentials (i) and (iii), we have also solved the problem considering only contribution (ii) and neglecting (i) and (iii). For this case, in the bound state condition, instead of r_-, we have $r'' = (b+5)/(b+2)^2$. As expected, there is still a solution $b'' \simeq 0.00639$.

Hence, we obtain $b' < b'' < \bar{b}$, and our analysis shows: that the attractive part in the effective potential in cases (i) and (iii) dominates its repulsive part; that the attraction associated with case (ii), i.e. the exchange of a one quasi-meson term, substantially increases the binding.

The graphs of the curves for that enter in the above bound state conditions are depicted in Figure 1 below.

In order to get a yet better understanding on the balance between the attractive and the repulsive parts, we analyze the effect of gauge correlations in case (iii). Namely, instead of using the gauge correlation integral

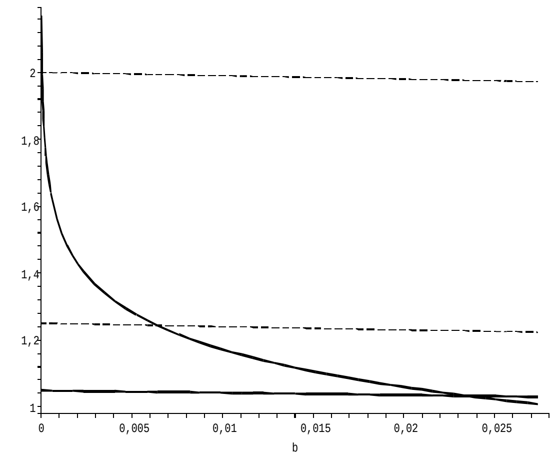

FIG. 1: Bound state condition graphs, as functions of b. The solid curve diverging at b=0 is r_{00}. The lower almost horizontal solid curve is r_-. The upper dashed curve is r' and the lower one is r''. The b coordinate for the intersection points with r_{00} are as follows. For r_-, $b = \bar{b} \simeq 0.02359$; for r', $b = b' \simeq 0.00006$; for r'', $b = b'' \simeq 0.00639$.

$\mathcal{I}_4$ of Eq. (A6), we erroneously use the uncorrelated contribution $(\mathcal{I}_2)^2$. The effect of doing so is to replace the operator $(P/2 - P_1)$ in R of Eq. (12) by the identity, in term (iii). With this, there is no more recombination of the terms (i) and (iii) and, consequently, both terms remain of order κ^0. When we perform the rescaling by the factor $\kappa^2/2$ [see Eq. (9)], the potential associated with these two terms acquires a multiplicative factor $2/\kappa^2$. As $0 < \kappa << 1$, this means a large coupling coefficient. To see the effect of the uncorrelated gauge fields, we solve the problem again. Here, we obtain a bound state condition like Eq. (17) with r_- replaced by $1/(b-1)$, in the $\kappa \to 0$ limit. As we have the rigorous bound $r_{00} > 1/(b+2)$ ($K(a)$ is monotone increasing!), we see that there is no bound state solution even though we are including the attractive quasi-meson exchange contribution.

We can understand this in the context of a lattice Schrödinger operator. We consider the Schrödinger operator, in $\ell_2(\mathbb{Z}^2)$, $(-\Delta/2)$ minus the potential $V_0 + \lambda V_1$, where V_0 is a repulsive potential at the origin that we send to $+\infty$ (corresponding to taking $\kappa \to 0$), giving rise to a Dirichlet boundary condition at the origin. λV_1 is a range-one attractive potential. The bound state condition we obtain for this case is given in Eq. (17), but with r_- replaced by $\check{r} \equiv [b + 2(1 - 1/\lambda)]^{-1}$. For positive λ, this equation has a bound state solution only for λ above a critical value $\check{\lambda}$. A numerical analysis shows that $\check{\lambda}$ is roughly of order $\check{\lambda} \simeq 2.5$.

Although the exchange of one quasi-meson gives rise to an attractive potential, our analysis makes clear that the correlation with four gauge fields $\mathcal{I}_4$ is the main mechanism in the formation of meson-meson bound states, at least in the zero total isospin sector that we analyze here.

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