Excitation Spectrum and Staggering Transformations in Lattice Quantum Models

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We consider the energy-momentum excitation spectrum of diverse lattice Hamiltonian operators: the generator of the Markov semi-group of Ginzburg-Landau models with Langevin stochastic dynamics, the Hamiltonian of a scalar quantum field theory and the Hamiltonian associated with the transfer matrix of a classical ferromagnetic spin system at high temperature. The low-lying spectrum consists of a one-particle state and a two-particle band. The two-particle spectrum is determined using a lattice version of the Bethe-Salpeter equation. In addition to the two-particle band, depending on the lattice dimension and on the attractive or repulsive character of the interaction between the particles of the system, there is, respectively, a bound state below or above the two-particle band. We show how the existence or non-existence of these bound states can be understood in terms of a non-relativistic single particle lattice Schrödinger Hamiltonian with a delta potential. A staggering transformation relates the spectra of the attractive and the repulsive cases.

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1. INTRODUCTION

In previous works [1–3], we determined some interesting features of the low-lying energy-momentum ($e-m$) spectrum associated with the generator of stochastic Ginzburg-Landau (G-L) continuous time, continuous and unbounded spin $\mathbb{Z}^d$ infinite space lattice models of non-equilibrium statistical mechanics. The generator or Hamiltonian is that of an infinite lattice quantum field theory with slightly nonlocal interactions. The Hamiltonian spectrum has a quasi-particle interpretation with a one-particle isolated dispersion curve and a finite two-particle band. The interaction of the particles has an attractive or repulsive character depending on the parameters of the model. In the attractive case, for $d = 1, 2$, a bound state occurs below the two-particle threshold and, quite surprisingly, a bound state occurs above the band in the repulsive case. These results are established using a Feynman-Kac formula and a lattice version of the Bethe-Salpeter (B-S) equation, first, up to the ladder approximation and then, controlling perturbations about Gaussians, for the complete models.

Here we show that similar spectral phenomena occur for other diverse Hamiltonians, i.e. that of a lattice scalar quantum field theory and the Hamiltonian associated with minus the logarithm of the transfer matrix of classical ferromagnetic spin systems of equilibrium statistical mechanics at high temperature (small β) on $\mathbb{Z}^d$. In the case of the spin system, there is a discrete imaginary time local $\mathbb{Z}^{d-1}$ lattice field theory associated with the model and the complete model is not necessarily described by a perturbation of a Gaussian [4].

Furthermore, we show that the spectral results can be understood in terms of the spectrum of a one-particle lattice Schrödinger Hamiltonian $H = -\Delta + V(\lambda) \equiv H_0 + V(\lambda)$, where Δ is the lattice Laplacian and $V \equiv V(\lambda) = \lambda\delta$ is an attractive ($\lambda < 0$) or repulsive ($\lambda > 0$) delta potential at the origin. It turns out that the Schrödinger resolvent equation with V as a perturbation

$$(H - zI)^{-1} = (H_0 - zI)^{-1} - (H_0 - zI)^{-1}\, V\, (H - zI)^{-1} \quad (1)$$

is a good approximation to the zero system momentum B-S equation (see [7]) which has the general form

$$D \,=\, D^0 \,+\, D\, K\, D^0\,.$$

D^0 corresponds to the free resolvent, D to the interacting resolvent and K to minus the potential V. In the ladder approximation, the nonlocal K is a local delta potential in the models we consider. Now, H acting on $\ell^2(\mathbb{Z}^d)$ has a continuous bounded spectrum starting at zero and for $\lambda < 0$ a negative energy bound state. For $\lambda > 0$, there is a positive energy bound state above the continuum.

In contrast to the case of continuum quantum field theory, a richer low-lying $e - m$ spectrum for the lattice models we analyze is allowed by the absence of the Poincaré invariance on the lattice. Much of this spectral structure disappears in the continuum limit, which will not concern us here. We also point out that the above lattice Schrödinger operator can be understood in another way, i.e. the one-particle time-independent lattice Schrödinger eigenvalue equation with a delta potential, which corresponds to the normal mode equation of polarized classical oscillations of an infinite lattice equal-mass, Hookian spring system with an isotopic type defect (different mass) at the origin. The normal modes of such

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systems are discussed in [5, 6], in particular the physics for the modes below and above the continuum frequency band. The case of the repulsive potential corresponds to a small mass at the origin; there is a highest-energy normal mode (bound state), with energy above the continuous energy band, where adjacent particles are moving 180^o out of phase, in opposite directions.

The paper is organized as follows. In Section 2, we obtain the spectral properties of H, introduce the staggering transformation and analyze its relation with the spectrum. In Section 3, we briefly show how staggering applies to the lattice scalar quantum field model and to lattice stochastic G-L type models. In Section 4, we determine the spectrum above the two-particle band for a classical ferromagnetic spin system at high temperature and make the correspondence with the non-relativistic single-particle Schrödinger delta potential Hamiltonian.

2. DELTA FUNCTION POTENTIAL AND STAGGERING

The *e-m* spectral behavior around the two-particle band for the models described above can be understood in terms of the spectral properties of a $\mathbb{Z}^d$ lattice Schrödinger Hamiltonian H for a particle in a delta potential at the origin, i.e., with $\vec{x} = (x^1, \ldots, x^d) \in \mathbb{Z}^d$,

$$H = H_0 + V \equiv -\Delta + V \quad ; \quad V(\vec{x}\,) = \lambda\delta(\vec{x}\,)\,, \tag{2}$$

where H acts on $\ell_2(\mathbb{Z}^d)$, and ($\mathbf{e}^j$ is the unit vector along the j th direction) $-\Delta f(\vec{x}\,) = 2df(\vec{x}\,) - \sum_{j=1}^d f(\vec{x} + \mathbf{e}^j) - \sum_{j=1}^d f(\vec{x} - \mathbf{e}^j)$, $f \in \ell_2(\mathbb{Z}^d)$. In momentum space, H_0 is multiplication by $2\sum_{j=1}^d \left(1 - \cos p^j\right) \equiv -\tilde{\Delta}(\vec{p})$, $\vec{p} = (p^1, \ldots, p^d) \in \mathbf{T}^d \equiv (-\pi, \pi]^d$, and the spectrum of H_0 is $[0, 4d]$, and is absolutely continuous. Unlike the continuum, the lattice delta potential is a bounded operator for any d.

We now determine the spectral properties of the Hamiltonian H of Eq. (2). First, we define a unitary *staggering* transformation U, which plays a key role in understanding the relation between the spectrum of H with $\lambda > 0$ and $\lambda < 0$, by

$$Uf(\vec{x}\,) = (-1)^{\sum_{j=1}^d x^j}\; f(\vec{x}\,) \quad ; \quad f \in \ell_2(\mathbb{Z}^d)\,, \tag{3}$$

and satisfies $U^2 = I$, $U^{-1} = U$, and its momentum representation has the form $(Uf)^{\sim}(\vec{p}\,) = \tilde{f}(\vec{\pi} - \vec{p}\,)$, where $\vec{\pi} \equiv (\pi, \pi, \ldots, \pi)$, and $\tilde{f}(\vec{p}\,) = \sum_{\vec{x}\in\mathbb{Z}^d} e^{-i\vec{p}.\vec{x}} f(\vec{x})$. U takes smooth functions into rough functions and vice-versa and has the important intertwining property

$$-\Delta + \lambda\delta = U[-1(-\Delta - \lambda\delta - 4d)]U^{-1}\,. \tag{4}$$

If E is a point in the spectrum of H, with corresponding eigenfunction ψ for an attractive potential ($\lambda < 0$), then $-E + 4d$ and $U\psi$ are the corresponding eigenvalue and eigenfunction of H with a repulsive potential ($\lambda > 0$). Thus, it is enough to consider the familiar attractive case; the repulsive case (unlike that of the $d = 1$ continuum Hamiltonian) exhibits unusual spectral properties.

We now determine the spectral properties of H for $\lambda < 0$. The resolvent $(H - z)^{-1}$ contains all the spectral information of H. For $z \in \mathbb{C}$, $z \notin [\sigma(H) \cap \sigma(H_0)]$, where $\sigma(A)$ denotes the spectrum of A, solving Eq. (1), we obtain

$$(H - z)^{-1}(\vec{x}, \vec{y}) = (H_0 - z)^{-1}(\vec{x}, \vec{y}) - (H_0 - z)^{-1}(\vec{x}, \vec{0}) \times \frac{\lambda}{1 + \lambda(H_0 - z)^{-1}(\vec{0}, \vec{0})}\,(H_0 - z)^{-1}(\vec{0}, \vec{y})\,, \tag{5}$$

and $(H_0 - z)^{-1}(\vec{x}, \vec{y}) = \frac{1}{(2\pi)^d}\int_{\mathbf{T}^d} e^{i\vec{p}.(\vec{x}-\vec{y})}[-\tilde{\Delta}(\vec{p}) - z]^{-1}\, d\vec{p}$, for $z \notin [0, 4d]$. Outside $[0, 4d]$, the spectrum of H arises from z singularities of Eq. (5), that can only occur as zeroes of the denominator, i.e. for $z = -E_b$, $E_b > 0$, we have

$$\lambda(H_0 - z)^{-1}(\vec{0}, \vec{0}) = \frac{\lambda}{(2\pi)^d}\int_{\mathbf{T}^d} \frac{d\vec{p}}{-\tilde{\Delta}(\vec{p}) + E_b} = -1\,. \tag{6}$$

Hence, since $-\tilde{\Delta}(\vec{p}) \approx |\vec{p}|^2$, for small $|\vec{p}|$, there is a unique bound state $-E_b$, for $d = 1, 2$ and any $\lambda < 0$. For $d \geq 3$, the integral in Eq. (6) converges for $E_b = 0$, so there is a critical value $\lambda_c < 0$ for the occurrence of a bound state similar to the continuum case (see the Birman-Schwinger bound in [8]) .

From the Perron-Frobenius theorem for e^{-H} (see [8]), the associated ground state eigenfunction $\psi(\vec{x})$ is positive (i.e., $\psi(\vec{x}) > 0$, for all $\vec{x}$) and, using the spectral theorem, the bound state eigenfunction is $\psi(\vec{x}) = [\lim_{z\nearrow -E_b}\left(-(-E_b) - z\right)(H - z)^{-1}(\vec{x}, \vec{x})]^{1/2}$. Moreover, expanding $1 + \lambda(H_0 - z)^{-1}(\vec{0}, \vec{0})$ about $z = -E_b$ and up to a normalization $\mathcal{N}$, we find $\psi(\vec{x}) = \mathcal{N}\int_{\mathbf{T}^d} e^{i\vec{p}.\vec{x}}[-\tilde{\Delta}(\vec{p}) + E_b]^{-1}\, d\vec{p}$, which is even. Also, by the Payley-Wiener theorem (see [9]) $\psi(\vec{x})$ decays exponentially. This completes the description of H for $\lambda < 0$. By Eq. (4), the spectrum of H for $\lambda > 0$ is $(4d + E_b) \cup [0, 4d]$. The surprising feature is the existence of a bound state above the continuous spectrum at $E_a \equiv 4d + E_b$. As the eigenfunction for the attractive case is positive, the repulsive case bound state eigenfunction has maximum oscillation, by Eq. (3).

For comparison with the ladder approximation to the lattice B-S equation in the ensuing sections, it is convenient to have the momentum space form of Eq. (5). With $H_0 = -a\Delta$, and with $a > 0$ introduced here for later use, it reads

$$(H - z)^{-1\sim}(\vec{p}, \vec{q}) = \frac{(2\pi)^d}{-a\tilde{\Delta}(\vec{p}) - z}\,\delta(\vec{p} + \vec{q}) - \frac{1}{-a\tilde{\Delta}(\vec{p}) - z} \times \frac{\lambda}{1 + \frac{\lambda}{(2\pi)^d}\int_{\mathbf{T}^d} \frac{d\vec{u}}{-a\tilde{\Delta}(\vec{u}) - z}}\,\frac{1}{-a\tilde{\Delta}(\vec{q}) - z}\,. \tag{7}$$

3. QUANTUM FIELD THEORY AND STOCHASTIC GENERATOR HAMILTONIAN

In this section, we derive spectral results for the e-m spectrum of scalar lattice quantum field models, and relate the ladder B-S equation to the non-relativistic lattice one-particle Schrödinger resolvent equation of Section 2, and show how the appearance or absence of bound states below and above the two-particle band can be understood in terms of this non-relativistic model.

Our analysis of the scalar quantum field model is patterned after [1–3]. Here, the discrete space formal Hamiltonian is

$$\begin{aligned} H \; = \; & \sum_{\vec{x}\in\mathbb{Z}^d} \left\{ -\frac{1}{2}\frac{\partial^2}{\partial\varphi(\vec{x})^2} + \frac{1}{2}\varphi(\vec{x})\left[(-\Delta+m^2)\varphi\right](\vec{x}) \right. \\ & \left. +\lambda_6 : \varphi(\vec{x})^6 : +\lambda : \varphi(\vec{x})^4 : \right\} , \end{aligned}$$

where $\lambda_6 > 0$ and λ can be either positive (repulsive case) or negative (attractive case). $m > 0$ is fixed and large, and λ_6 and $|\lambda|$ are taken sufficiently small. The analysis of the two-point function $S_\lambda(x)$ shows there is an isolated dispersion curve $E_\lambda(\vec{p}) = [-\tilde{\Delta}(\vec{p})+m^2]^{1/2}+\mathcal{O}(\lambda^2)$, $\vec{p}\in\mathbf{T}^d$. There is also a two-particle band with upper and lower envelopes given by $E^{\pm}_\lambda(\vec{p}) = 2[\sum_{j=1}^d 2(1\pm\cos\frac{p^j}{2}) + m^2]^{1/2} + \mathcal{O}(\lambda^2)$, where $\vec{p}$ is the total spatial field momentum and for the lower envelope each particle has momentum $\vec{p}/2$. Outside the band and below the four-particle threshold, a B-S analysis in the ladder approximation shows there is a bound state if, for $\lambda' = (2\pi)^{-2(d+1)}\lambda$ and $\tilde{S}_\lambda$ being the Fourier transform of S_λ, for $q = (q^0,\vec{q})\in\mathbb{R}\times\mathbf{T}^d$,

$$1 + 2\lambda' \int \tilde{S}_\lambda(k^0/2-q^0,\vec{q})\tilde{S}_\lambda(k^0/2+q^0,\vec{q})\,dq = 0 \; ,$$

for k^0 in the positive imaginary axis. We consider only mass spectrum here and in the sequel, i.e. energy-momentum spectrum at zero spatial field momentum. Keeping only the dominant one-particle contribution to $\tilde{S}_\lambda$ and performing the q^0 integration gives the condition $1 + \lambda'\int G(k^0,\vec{u})d\vec{u} = 0$, where $G(k^0,\vec{u}) = 2\pi d_\lambda(\vec{p})[E_\lambda(\vec{p})((k^0)^2+E_\lambda(\vec{p})^2)]^{-1}$ and $d_\lambda(\vec{p}) = 1+\mathcal{O}(\lambda^2)$. For small $|\vec{p}|$, $G(\vec{u},k^0)\approx \pi\left[4m^2(\vec{p}^2/m+\epsilon\right]^{-1}$ upon setting $k^0 = i(2m-\epsilon)$, for $\epsilon>0$ and small.

The correspondence with Eq. (7) is now clear upon taking $a = 1/m$ and $z = -\epsilon$. A similar result holds approaching the top of the first band from above, taking λ positive for the existence of the upper stable state. The absence of bound states for $d\geq 3$ is also in agreement with the non-relativistic behavior of Section 2. The exact intertwining relation of the non-relativistic Hamiltonian is here only approximate. However, it becomes exact, in the $m\to\infty$ limit.

A similar correspondence with the spectrum of the lattice one-particle Schrödinger Hamiltonian holds for the two-particle spectrum of the infinite spatial lattice stochastic G-L generator Hamiltonian

$$\begin{aligned} H \; = \; & \sum_{\vec{x}\in\mathbb{Z}^d} \left\{ -\frac{1}{2}\frac{\partial^2}{\partial\varphi(\vec{x})^2} + \frac{1}{8}\varphi(\vec{x})[(-\Delta+m^2)^2\varphi](\vec{x}) \right. \\ & +\frac{\lambda}{4}[(-\Delta+m^2)\varphi](\vec{x})\mathcal{P}'(\varphi(\vec{x})) \\ & \left. + \left[\frac{\lambda^2}{8}\mathcal{P}'(\varphi(\vec{x}))^2 - \frac{\lambda}{4}\mathcal{P}''(\varphi(\vec{x})) - \frac{(2d+m^2)}{4}\right] \right\} . \end{aligned}$$

considered in [1–3]. Here $m>0$ is large, $0<\lambda<<1$ and $\mathcal{P}$ is an even polynomial of degree $2N$, bounded from below and starting with a quartic term. For some $d = 1,2$, the bound state below the band for the attractive case is related by staggering to the bound state above the band for the repulsive case.

4. CLASSICAL FERROMAGNETIC SPIN SYSTEMS AT HIGH TEMPERATURE

In this section, we determine new excitation spectrum for the transfer matrix associated with ferromagnetic classical spin systems on a $\mathbb{Z}^d$ lattice, and at high temperature (small $\beta>0$). Also, we show how the spin system spectral properties can be understood in the context of the delta function Hamiltonian of Section 2.

Formally, the partition function for the system is

$$Z = \int e^{\beta\sum\, s(\vec{x})s(\vec{y})} \prod_{\vec{u}\in\mathbb{Z}^d} d\mu\left(s(\vec{u})\right) \; ,$$

where $s(\vec{x})\in\mathbb{R}$ is the spin variable at the lattice site $\vec{x}\in\mathbb{Z}^d$; the sum runs over unordered nearest neighbor pairs of sites $\vec{x}$ and $\vec{y}$, and $d\mu(s)$ is the single site spin distribution (ssd) given by $d\mu(s) = e^{-V(s)}\,ds$, which is taken to be even. The normalized kth-order moment of the ssd is denoted by $\left\langle s^k\right\rangle$.

In previous works [4] on the spectrum of the transfer matrix it is shown that there is an associated discrete imaginary-time, $\mathbb{Z}^{d-1}$ space lattice quantum field theory. The Hilbert space $\mathcal{H}$ and self-adjoint $e-m$ operators $H\geq 0$, P^j, $j=1,\ldots,d$, are constructed from correlation functions via a Feynman-Kac formula (see [10]). In [4], it is shown that the low-lying $e-m$ spectrum has a particle interpretation. There is a one-particle state with the isolated dispersion curve $E(\vec{p})$, $\vec{p}\in\mathbf{T}^{d-1}$, with $E(\vec{p}) \geq E(\vec{0}) \equiv m(\beta)$, where $E(\vec{p}) = -\ln\beta\left\langle s^2\right\rangle - 2\beta\left\langle s^2\right\rangle(d-1) + 2\beta\left\langle s^2\right\rangle\sum_{j=1}^{d-1}\left(1-\cos p^j\right) + \mathcal{O}(\beta^2)$. Furthermore, in [4], using an 'equal-time', relative coordinate B-S equation, it is shown that the mass spectrum up to the two-particle threshold $2m(\beta)$ depends on the ssd in the following way. If the Gaussian domination inequality [11] for the ssd holds, i.e. if $\zeta\equiv\left\langle s^4\right\rangle - 3\left\langle s^2\right\rangle^2 < 0$, there is no mass spectrum in $(m(\beta),2m(\beta))$. On the other hand, if $\zeta>0$, there is a bound state with mass M_b given by $M_b = 2m(\beta) - |\ln(1-\Upsilon)| + \mathcal{O}(\beta)$, where $\Upsilon = [\left\langle s^4\right\rangle - 3\left\langle s^2\right\rangle^2]/[\left\langle s^4\right\rangle - \left\langle s^2\right\rangle^2]$. Besides, M_b is the only point in the mass spectrum in $(m(\beta),2m(\beta))$.

Here, we obtain spectral results up to near the four-particle threshold $4m(\beta)$, using a ladder approximation to the B-S equation. Above $2m(\beta)$, we show that the mass spectrum forms a band from $2m(\beta)$ to $2m(\beta)+w$, where $w = 2E(\vec{p}=\vec{\pi}) - 2E(\vec{0}) = 8(d-1)\left\langle s^2\right\rangle\beta + \mathcal{O}(\beta^2)$ denotes the first band width. In the interval from $2m(\beta)+w$ to $4m(\beta)-\epsilon$, $\epsilon \equiv \epsilon(\beta) \searrow 0$ as $\beta \searrow 0$, i.e. near the four-particle threshold, we show the following: for any d, if $\zeta < 0$, there is a state with mass $M_u = 2m(\beta)+w+|\ln(1-\Upsilon)|+\mathcal{O}(\beta)$ and M_u is the only point in the spectrum in $(2m(\beta)+w, 4m(\beta)-\epsilon)$; if $\zeta > 0$, there is no mass spectrum in $(2m(\beta)+w, 4m(\beta)-\epsilon)$.

In the ladder approximation, with $\rho = \Upsilon/[2\left\langle s^2\right\rangle^2]$, the condition for the existence of a bound state outside the band is $1-(2\pi)^{-d}\rho\int_{-\infty}^{\infty}\int_{\mathbf{T}^{d-1}} 2\tilde{S}(u^0,\vec{p})\tilde{S}(k^0-u^0,\vec{p})du^0\,d\vec{p} = 0$, for k^0 on the positive imaginary axis and outside the band.

The spectral representation for the two-point function is given by $S(x) = \int_0^{\infty}\int_{\mathbf{T}^{d-1}} e^{-E|x^0|}e^{i\vec{p}.\vec{x}}\,d\sigma_{\vec{p}}(E)\,d\vec{p}$ where $d\sigma_{\vec{p}}(E) = z(\vec{p},\beta)\,\delta(E-E(\vec{p}))\,dE + d\hat{\sigma}_{\vec{p}}(E)$ with $d\sigma_{\vec{p}}(E)$ and $d\hat{\sigma}_{\vec{p}}(E)$ being positive measures; the support of $d\hat{\sigma}_{\vec{p}}(E)$ is contained in $(3m(\beta)-\bar{\epsilon},\infty)$ and has the bound $\hat{\sigma}_{\vec{p}}(0,\infty) = \int_0^{\infty} d\hat{\sigma}_{\vec{p}}(E) = \mathcal{O}(\beta)$. Furthermore, $z(\vec{p},\beta) = \left\langle s^2\right\rangle(2\pi)^{1-d} + \mathcal{O}(\beta)$. Maintaining only the product of one-particle contributions to $\tilde{S}$, the bound state condition becomes $(2\pi)^{-2(d-1)}\rho\int_{\mathbf{T}^{d-1}} H(\vec{p},k^0)\,d\vec{p} = 1$, where

$$H(\vec{p},k^0) = 2(2\pi)^{(d-1)}\left\langle s^2\right\rangle^2 \frac{\sinh 2E(\vec{p})}{\cosh 2E(\vec{p}) - \cosh k^0}\,. \quad (8)$$

Upon setting $k^0 = i\chi$, $\chi = 2m(\beta)-\epsilon_b$, $\epsilon_b > 0$, and after using $e^{E(\vec{p})} = \beta^{-1}\left\langle s^2\right\rangle^{-2} - 2\sum_{j=1}^{d-1}\cos p^j + \mathcal{O}(\beta)$ in Eq. (8), gives for $\beta = 0$, $\Upsilon\,[1-e^{-\epsilon_b}]^{-1} = 1$ and $-\epsilon_b = \ln[(\zeta + 2\left\langle s^2\right\rangle^2)/(2\left\langle s^2\right\rangle^2)]$. In particular Υ and hence ζ must be positive.

On the other hand, approaching the band from above we write $\chi = 2m(\beta)+w+\epsilon_u$, $\epsilon_u > 0$, which leads to the condition, for $\beta = 0$, $\Upsilon\,[1-e^{\epsilon_u}]^{-1} = 1$ and $\epsilon_u = \ln[(2\left\langle s^2\right\rangle^2)/(\zeta + 2\left\langle s^2\right\rangle^2)]$, for the existence of a state. Thus, Υ and ζ are negative, and we have shown our result.

We now make the connection between the B-S equation and the Schrödinger Hamiltonian resolvent equation of Section 2. Setting $k^0 = i[2m(\beta)+\beta z']$ we have, for small β and taking into account the form of $E(\vec{p})$, $H(\vec{p},k^0) \simeq 2(2\pi)^{d-1}\left\langle s^2\right\rangle^2 [2\beta\left\langle s^2\right\rangle^2(-\tilde{\Delta}(\vec{p})) - \beta z']^{-1}$, so that the bound state condition becomes

$$1 - \rho\int_{\mathbf{T}^{d-1}} \frac{d\vec{u}}{2\beta\left\langle s^2\right\rangle\,(-\tilde{\Delta}(\vec{u})) - \beta z'} = 0\,. \quad (9)$$

Comparing with the $d-1$ resolvent of Eq. (7) of Section 2, we make the identification of the coupling constant $\lambda \approx \rho$, spectral parameter $z = \beta z'$, and the free Hamiltonian $H_0 = 2\left\langle s^2\right\rangle\beta(-\Delta)$. Note that, although the integral in Eq. (9) is finite for $d \geq 3$, the factor in front is ρ/β which is arbitrarily large as $\beta \searrow 0$. For this reason, a bound state exists for all values of d if ρ and ζ are positive and β is sufficiently small. We remark that this behavior agrees with the results of Section 2.

5. CONCLUSIONS

We showed how staggering transformations play an important role in the understanding of the low-lying spectrum of diverse quantum lattice models.

It would be relevant to extend the present analysis and consider the role played by staggering transformations in multi-phase regions and exactly soluble models, as well as to investigate degeneracies in multi-component cases and the existence of soliton-anti-soliton solutions for space lattice classical nonlinear wave equations and their quantum analogues.

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