Understanding baryons from first principles

Paulo A. Faria da Veiga* and Michael O'Carroll†
Departamento de Matemática, ICMC-USP, C.P. 668, 13560-970 São Carlos, SP, Brazil.

Ricardo Schor‡
Departamento de Física-ICEx, UFMG, C.P. 702, 30161-970 Belo Horizonte MG, Brazil

(Dated: October 29, 2002)

We obtain an understanding of baryons and anti-baryons in the strong coupling regime ($\kappa <<$ 1, where κ is the hopping parameter) of Euclidean lattice QCD. It is shown that they arise as tightly bound, bound states of three (anti-)quarks. The appearance of each of these particles is manifested by the occurence of an isolated dispersion curve (upper gap property) in the energy-momentum spectrum, with asymptotic mass of order $-3\ln\kappa$. The upper gap property holds in the full gauge invariant (anti-)baryon space of states, at least up to $-4\ln\kappa$. Besides, we establish a spectral representation for the two-point baryon correlation function as well as determine symmetry properties. To capture the essence of the mechanism of baryon formation with the minimum of algebraic complexity we consider the single flavor case in $2+1$ space-time dimensions and two-dimensional spin matrices.

PACS numbers: 02.30.Tb, 11.15.Ha, 11.10.St, 24.85.+p, 12.38.-t

The existence and spectral properties of quantum field theories arise as fundamental problems in physics. In quantum chromodynamics (QCD), it is fundamental to establish on a rigorous basis the low-energy momentum spectrum of particles and their bound states, in particular, to prove the existence of mesons and baryons and their bound states. Spectral properties of this sort involve low-energies and the standard way to study them is to use a lattice regularization of the continuum. Long distance properties should not be affected too much by this procedure. Our understanding of confinement comes in this way, and in fact was a reason for the introduction of lattice QCD by Wilson (see [1, 2] and [3, 4] for a recent review).

As our main result, we obtain an understanding of baryons from first principles by showing that they arise as tightly bound, bound states of three quarks and their occurrence is manifested by the appearance of an isolated dispersion curve in the energy-momentum (e-m) spectrum. We adopt a lattice regularization and obtain a qualitative picture of the spectrum from first principles in the strong coupling regime corresponding to quark confinement. No approximations are made and in this context our results are exact. More specifically, we consider lattice QCD in the Euclidean formulation as in [2, 5], with small hopping parameter (strong coupling) and large glueball mass (see [6]) in the baryon Fermi sector. For the Hamiltonian approach to baryon masses see [7, 8]. For the determination of masses for the meson sector, in the Euclidean approach, see [9].

We want to capture the essence of the mechanism of baryon formation. Having this goal in view, and in order to reduce algebraic complexity, we consider the single flavor case in $2+1$ space-time dimensions with two-dimensional Dirac matrices, so that there is no internal spin.

Here, we show that there is a (baryon) particle and an anti-particle manifested by an isolated dispersion curve in the e-m spectrum (upper gap) and much more: it is the *only* spectrum in the full space of states up to near the two-meson threshold, so that the upper gap persists. Also, analogous to the Källen-Lehman representation in quantum field theory, we obtain a spectral representation for the two-particle correlation function which allows us to relate the singularities of its Fourier transform to the e-m spectrum via a Feynman-Kac formula. Furthermore, we analyze the symmetries of the model.

We emphasize that the upper gap property has not been established in the Hamiltonian formulation treatments.

The determination and control of the baryonic mass spectrum is an essential step towards the understanding of e.g. baryon-baryon bound states (such as for the nuclear force) from first principles and thus bridging the gap between QCD and the effective baryon, effective meson picture of nuclear forces through a single and multiple boson exchange arising from a Yukawa interaction (see [10, 11]). For the determination of bound state spectrum in spin and stochastic Ginzburg-Landau lattice systems see [12] and [13].

We now define the gauge-matter model and show how our results are obtained. The quantum mechanical Hilbert space and e-m operators are obtained by a standard construction from the thermodynamic limit of gauge invariant correlation functions of a statistical mechanical model where the Fermionic degrees of freedom are described by Grassmann variables (see [2, 5, 14] for details). The partition function of the model is given

*Electronic address: veiga@icmc.usp.br
†Electronic address: ocarroll@icmc.usp.br
‡Electronic address: rsschor@fisica.ufmg.br

formally by

$$Z = \int e^{-S(\psi,\bar{\psi},g)}\, d\psi\, d\bar{\psi}\, d\mu(g) \ . \tag{1}$$

and for $F(\bar{\psi},\psi,g)$, the normalized expectations are denoted by $\langle F\rangle$.

Labelling by 0 the time direction, the unit lattice is taken as $\mathbb{Z}^3_o$, where $u = (u^0,\vec{u}) = (u^0,u^1,u^2) \in \mathbb{Z}^3_o \equiv \mathbb{Z}_{1/2}\times\mathbb{Z}^2$, where $\mathbb{Z}_{1/2} = \{\pm 1/2, \pm 3/2, \ldots\}$, so that, in the continuum limit, two-sided equal time limits of Fermi fields correlations can be accommodated. At each site $u \in \mathbb{Z}^3_o$, there are Fermion Grassmann fields $\psi_{\alpha,a}(u)$, associated with a quark, and $\bar{\psi}_{\alpha,a}(u)$, associated with an anti-quark, which carry a Dirac spin index $\alpha = 1,2 \equiv +,-$ and an $SU(3)$ color index $a = 1,2,3$. Also, letting e^μ, $\mu = 0,1,2$, denote the unit lattice vector for the μ-direction, for each nearest neighbor oriented lattice bond $< u, u\pm e^\mu >$ there is an $SU(3)$ matrix $U(g_{u,u\pm e^\mu})$ parametrized by the gauge group element $g_{u,u\pm e^\mu}$ and satisfying $U(g_{u,u+e^\mu})^{-1} = U(g_{u+e^\mu,u})$. Associated with each lattice oriented plaquette p there is a plaquette variable $\chi(U(g_p))$ where $U(g_p)$ is the orientation-ordered product of matrices of $SU(3)$ of the plaquette oriented bonds, and χ is the trace. The model action is given by

$$S(\psi,\bar{\psi},g) = \frac{\kappa}{2}\sum \bar{\psi}_{\alpha,a}(u)\Gamma^{\epsilon e^\mu}_{\alpha\beta}(g_{u,u+\epsilon e^\mu})_{ab} \times$$
$$\psi_{\beta,b}(u+\epsilon e^\mu) + \sum_{u\in\mathbb{Z}^3_o} \bar{\psi}_{\alpha,a}(u) M_{\alpha\beta}\psi_{\beta,a}(u) - \frac{1}{g_0^2}\sum_p \chi(g_p), \tag{2}$$

where the first sum runs over $u \in \mathbb{Z}^3_o$, $\epsilon = \pm 1$ and $\mu = 0,1,2$. For notational simplicity, we drop U from $U(g)$, $m > 0$, $\kappa > 0$ is the quark-gauge coupling, $g_0 > 0$ describes the pure gauge strength and $M = m + 2\kappa$. Also, within the family of actions of [2], we have $\Gamma^{\pm e^\mu} = -1 \pm \gamma_\mu$, γ_μ being the hermitian traceless anti-commuting Pauli matrices σ_z, σ_x, σ_y, for $\mu = 0,1,2$, respectively. $d\mu(g)$ is the product measure over bonds of normalized $SU(3)$ Haar measures and the integrals over Grassmann fields are defined according to [14]. For a polynomial in the Grassmann variables with coefficients depending on the gauge variables, the fermionic integral is defined as the coefficient of the monomial of maximum degree, i.e. of $\prod_{u,\alpha,a}\bar{\psi}_{\alpha,a}(u)\psi_{\alpha,a}(u)$. In Eq. (1), $d\psi\, d\bar{\psi}$ means $\prod_{u,\alpha,a} d\psi_{\alpha,a}(u)\, d\bar{\psi}_{\alpha,a}(u)$ such that, with a normalization $\mathcal{N}_1 = \langle 1\rangle$, we have $\langle \psi_{\alpha,a}(x)\, \bar{\psi}_{\beta,b}(y)\rangle = (1/\mathcal{N}_1)\int \psi_{\alpha,a}(x)\, \bar{\psi}_{\beta,b}(y)\, e^{-\sum_u \bar{\psi}_{\alpha,a}(u)M_{\alpha\beta}\psi_{\beta,a}(u)}\, d\psi\, d\bar{\psi} = M^{-1}_{\alpha,\beta}\, \delta_{ab}\delta(x-y)$, with a Kronecker delta for space-time coordinates. With our restrictions on the parameters, there is a quantum mechanical Hilbert space of physical states (see below), for $\kappa > 0$; and the condition $m > 0$ guarantees that the one-particle free Fermion dispersion curve increases in each positive momentum component.

For small enough couplings κ and g_0^{-2}, by polymer expansion methods (see [2, 15]), the thermodynamic limit of correlations exists and truncated correlations have exponential tree decay. The limiting correlation functions are lattice translational invariant. Furthermore, the correlation functions extend to analytic functions in the coupling parameters.

We now recall the definition of the quantum mechanical Hilbert space $\mathcal{H}$ and the e-m operators starting from gauge invariant correlation functions, with support restricted to $u^0 = 1/2$. Letting $T_0^{x^0}$, $T_i^{x^i}$, $i = 1,2$, denote translation of the functions of Grassmann and gauge variables by $x^0 \geq 0$, $x \in \mathbb{Z}^3$; and for F and G only depending on coordinates with $u^0 = 1/2$, we have the Feynman-Kac (F-K) formula

$$(G, T_0^{x^0}T_1^{x^1}T_2^{x^2}F)_{\mathcal{H}} = \langle [T_0^{x^0}T_1^{x^1}T_2^{x^2}F]\Theta G\rangle \ ,$$

where Θ is an anti-linear operator which involves time reflection. Following [2], with the usual sum convention, the action of Θ on single fields is given by

$$\Theta\bar{\psi}_{\alpha,a}(u) = (\gamma_0)_{\alpha\beta}\psi_{\beta,a}(tu)\,,$$
$$\Theta\psi_{\alpha,a}(u) = \bar{\psi}_{\beta,a}(tu)(\gamma_0)_{\beta\alpha}\,;$$

where $t(u^0,\vec{u}) = (-u^0,\vec{u})$, for A and B monomials, $\Theta(AB) = \Theta(B)\Theta(A)$; and for a function of the gauge fields $\Theta f(\{g_{uv}\}) = f^*(\{g_{(tu)(tv)}\})$, $u,v \in \mathbb{Z}^3_o$, where $*$ means complex conjugate. Θ extends anti-linearly to the algebra. We do not distinguish between Grassmann, gauge variables and their associated Hilbert space vectors in our notation. As linear operators in $\mathcal{H}$, T_μ, $\mu = 0,1,2$, are mutually commuting; T_0 is self-adjoint, with $-1 \leq T_0 \leq 1$, and $T_{j=1,2}$ are unitary, so that we write $T_j = e^{iP^j}$ and $\vec{P} = (P^1,P^2)$ is the self-adjoint momentum operator, with spectral points $\vec{p} \in \mathbf{T}^2 \equiv (-\pi,\pi]^2$. Since $T_0^2 \geq 0$, we define the energy operator $H \geq 0$ by $T_0^2 = e^{-2H}$. More precisely, the positivity condition $\langle F\Theta F\rangle \geq 0$ is established in [2] but there may be nonzero F's such that $\langle F\Theta F\rangle = 0$. The collection of such F's is denoted by $\mathcal{N}$. Thus, a pre-Hilbert space $\mathcal{H}'$ can be constructed from the inner product $\langle G\Theta F\rangle$. The physical Hilbert space $\mathcal{H}$ is defined as the completion of the quotient space $\mathcal{H}'/\mathcal{N}$.

We now turn to our results. We restrict our attention to the subspace $\mathcal{H}_o \subset \mathcal{H}$ generated by an odd number of $\hat{\psi} = \bar{\psi}$ or ψ, for $g_0^{-2} << \kappa$. For the pure gauge case and small g_0^{-2}, the low-lying glueball spectrum is found in [6]. For large g_0, the glueball mass is $\approx 8\ln g_0$. For the even subspace, quark-anti-quark mesons are treated in [9].

We now state our result on the existence of particles. We consider the subspace $\mathcal{H}_3 \subset \mathcal{H}$ generated by vectors associated with Grassmann gauge invariant baryon-like fields given by, with ϵ_{abc} denoting the Levi-Civita symbol,

$$\hat{\phi}_\pm\left(u^0 = 1/2, \vec{u}\right) = \tfrac{1}{6}\epsilon_{abc}\hat{\psi}_{a\pm}\hat{\psi}_{b\pm}\hat{\psi}_{c\pm} = \hat{\psi}_{1\pm}\hat{\psi}_{2\pm}\hat{\psi}_{3\pm}\ ,$$

with either three ψ's or three $\bar{\psi}$'s. We show, for small enough $\kappa >> g_0^{-2}$, and for energies less than $-(4-\epsilon)\ln\kappa$, $0 < \epsilon << 1$, that in $\mathcal{H}_3$ the only e-m spectrum consists of dispersion curves $w_\pm(\vec{p})$, where

$$\begin{aligned} w_\pm(\vec{p}) = &\ 3\ln\tfrac{M}{2\kappa} + \ln[1 - 2\tfrac{\kappa^3}{M^3}(\cos p^1 + \cos p^2)] + \mathcal{O}(\kappa^4) \\ = &\ m_{\pm,\kappa} + \tfrac{\kappa^3}{M^3}|\vec{p}|^2 + \mathcal{O}(\kappa^4) \quad , \quad |\vec{p}| << 1\,; \end{aligned}$$

with $m_{\pm,\kappa} \equiv w_\pm(\vec{0})$ being the $\hat{\phi}_\pm$ baryon masses. Adapting the methods of [6, 15], it can be shown that $w_\pm(\vec{p}) + 3\ln\kappa$ is real analytic in κ, for $|\kappa|$ small, and in each p^i, for $|\mathcal{I}\mathrm{m}\ p^i|$ small. Furthermore, the dispersion curves $w_\pm(\vec{p})$ are increasing in each p^i, and convex for small $|\vec{p}|$. We remark that for free quarks, the system kinetic energy for the three-particle threshold has an $\mathcal{O}(\kappa)$ coefficient, rather than a κ^3 coefficient. We associate the curve $w_-(\vec{p})$ with $\bar{\phi}_-$ and call it a baryon field; similarly $w_+(\vec{p})$ is associated with the anti-baryon field ϕ_+. This terminology is justified later on when, by using a charge conjugation symmetry, we show that the dispersion curves are indeed identical.

To see how our results are obtained, associated to the particle and anti-particle are the normalized two-point correlation functions (χ here denotes the characteristic function)

$$\begin{aligned} G_-(u,v) &= \langle \phi_-(u)\bar{\phi}_-(v)\rangle\, \chi_{u^0\le v^0} \\ &\quad -\langle \bar{\phi}_-(u)\phi_-(v)\rangle\, \chi_{u^0> v^0} = G_-(u-v)\ , \\ G_+(u,v) &= \langle \bar{\phi}_+(u)\phi_+(v)\rangle\, \chi_{u^0\le v^0} \\ &\quad -\langle \phi_+(u)\bar{\phi}_+(v)\rangle\, \chi_{u^0> v^0} = G_+(u-v)\ . \end{aligned}$$

We note that by parity symmetry defined below, the $u^0 = v^0$ extensions of the $u^0 > v^0$ parts agree with the corresponding $u^0 = v^0$ definitions.

We now restrict our discussion to G_-, as the analysis of G_+ is similar, and we drop the subscript $-$ from the notation. From the F-K formula, $G(x) = -(\bar{\phi}_-(1/2,\vec{0}), T_0^{|x^0|-1} e^{i\vec{P}.\vec{x}}\bar{\phi}_-(1/2,\vec{0}))_{\mathcal{H}}\,\chi_{x^0>0} + [\vec{x}\to -\vec{x}]\,\chi_{x^0>0}$, taking the Fourier transform of G, and inserting the spectral representations for T_0, T_1 and T_2, we obtain the following spectral representation ($\tilde{G}(p)$ denotes the Fourier transform of $G(x \equiv u - v)$)

$$\begin{aligned} \tilde{G}(p) = \tilde{G}(\vec{p}) + (2\pi)^2 \textstyle\int_{-1}^1 \int_{\mathbf{T}^2} &\frac{2(\cos p^0 - \lambda^0)}{1+(\lambda^0)^2 - 2\lambda^0\cos p^0}\ \times \\ \delta(\vec{p}-\vec{\lambda}) d_{\lambda^0} d_{\vec{\lambda}} (\bar{\phi}_-(1/2,\vec{0}), &\mathcal{E}(\lambda^0)\mathcal{F}(\vec{\lambda})\bar{\phi}_-(1/2,\vec{0}))_{\mathcal{H}} \end{aligned}$$

where $\tilde{G}(\vec{p}) = \sum_{\vec{x}\in\mathbb{Z}^2} e^{-i\vec{p}.\vec{x}} G(x^0 = 0, \vec{x})$ and $\mathcal{F}(\vec{\lambda}) = \mathcal{F}_1(\lambda^1)\mathcal{F}_2(\lambda^2)$, and $\mathcal{E}(\lambda^0)$ (respectively, $\mathcal{F}_i(\lambda^i)$,$i = 1,2$) is the spectral family for the operator T_0 (respectively P^i). The existence of an isolated dispersion curve in the e-m spectrum occurs as a singularity of $\tilde{G}(p)$ on the imaginary p^0 axis, for fixed $\vec{p}$. (Our analysis excludes a singularity for $p^0 = \pi + iq^0$, q^0 real). Using a decoupling of hyperplane method (see [6, 15]), more precise bounds on $G(x)$ and the convolution inverse are now obtained. $|G(x)|$ is bounded by $\mathrm{const}\, e^{-3|\ln(\kappa)\,x^0| - 2|\ln(\kappa)\,\tilde{x}\,|}$ and thus $\tilde{G}(p)$ is analytic in p^0 in $|\mathcal{I}m\ p^0| < 3|\ln\kappa|$. The decay of the convolution inverse $\Gamma(x)$, of $G(x)$, for $|x| \ge 2$, is bounded by $\mathrm{const}\, e^{-4|\ln(\kappa)\,x^0| - 2|\ln(\kappa)\,\tilde{x}\,|}$, which implies that $\tilde{\Gamma}(p)$ is analytic in the larger region $|\mathcal{I}\mathrm{m}\ p^0| < -4\ln\kappa$, and that $\tilde{\Gamma}(p)^{-1}$ provides a meromorphic extension of $\tilde{G}(p)$ to this region. Furthermore, using the short distance behavior of $G(x)$, obtained by expanding in κ, namely $G(0) = -c^3 + \mathcal{O}(\kappa^8)$, $G(\pm e^0) = -8(\kappa/2)^3 c^6 + \mathcal{O}(\kappa^5)$ and $G(\pm e^i) = -(\kappa/2)^3 c^6 + \mathcal{O}(\kappa^5)$, $i = 1, 2$, where $c = 2/M$, we find $\tilde{G}(p) = -c^3[1 + 2(\kappa/2)^3 c^3(8\cos p^0 + \cos p^1 + \cos p^2)] + \mathcal{O}(\kappa^5)$ and $\tilde{\Gamma}(p) = -c^{-3}[1 - 2(\kappa/M)^3(8\cos p^0 + \cos p^1 + \cos p^2)] + \mathcal{O}(\kappa^5)$. The dispersion curve $w(\vec{p})$ arises as the solution of $\tilde{\Gamma}(p^0 = -iw(\vec{p}), \vec{p}) = 0$, and this shows the spectral results considering only the subspace $\mathcal{H}_3 \subset \mathcal{H}_o$. To extend the spectral results to all of $\mathcal{H}_o$, we adapt the subtraction method of [6].

We now establish lattice charge conjugation and parity symmetries. The equality of the dispersion curves $w_\pm(\vec{p})$ follows from the former and the parity symmetry is used to define an intrinsic (anti-)baryon parity. For the gauge-Grassmmanian field algebra, we define the charge conjugation transformation C by

$$\begin{aligned} C\psi_{\alpha,a}(u) = \bar{\psi}_{\beta,a}(u)\,(\gamma_2)_{\beta\alpha}\ ; \ C\bar{\psi}_{\alpha,a}(u) = (\gamma_2)_{\alpha\beta}\,\psi_{\beta,a}(u); \\ Cf(\{g_{uv}\}) = f(\{g^*_{uv}\})\,;\ C(AB) = C(B)C(A). \end{aligned}$$

where A and B are Grassmannian monomials. The transformation is extended by linearity to a general element.

The action of Eq. (2) is invariant under C, such as $CS = S$, and $\langle CF\rangle_0 = \langle F\rangle_0$, where $\langle\cdot\rangle_0$ denotes the expectation with the first term (hopping term) in the action S set equal to zero. From these two properties, it follows that $\langle CF\rangle = \langle F\rangle$. In showing $\langle CF\rangle_0 = \langle F\rangle_0$, we use the explicit formula for the integration of a class (gauge invariant) function with the $SU(3)$ Haar measure ([16]). Since $C\phi_-(u) = -i\bar{\phi}_+(u)$ and $C\bar{\phi}_-(u) = i\phi_+(u)$, we have $\langle\phi_-(u)\bar{\phi}_-(v)\rangle = -\langle\bar{\phi}_+(u)\phi_+(v)\rangle$. From the F-K formulae, for $x^0 \ge 0$,

$$\begin{aligned} (\phi_-, e^{-Hx^0} e^{i\vec{P}.\vec{x}}\phi_-)_{\mathcal{H}} &= \langle\phi_-(x^0+1/2,\vec{x})\bar{\phi}_-(-1/2,\vec{0})\rangle\,, \\ (\bar{\phi}_+, e^{-Hx^0} e^{i\vec{P}.\vec{x}}\bar{\phi}_+)_{\mathcal{H}} &= \langle\bar{\phi}_+(x^0+1/2,\vec{x})\phi_+(-1/2,\vec{0})\rangle. \end{aligned}$$

so that by the equality of the right sides, such as the equality of moments of the spectral measure, we have the equality $(\phi_-, \mathcal{E}(\lambda^0)\mathcal{F}(\vec{\lambda})\phi_-)_{\mathcal{H}} = (\bar{\phi}_+, \mathcal{E}(\lambda^0)\mathcal{F}(\vec{\lambda})\bar{\phi}_+)_{\mathcal{H}}$ which implies the equality of the spectrum; in particular, the dispersion curves are identical. Furthermore, C can be lifted to $\mathcal{H}$ as a unitary operator. This follows from $\langle (CF)(\Theta CF)e^{-S}\rangle_0 = \langle F\Theta F e^{-S}\rangle_0$; thus if $F\in\mathcal{N}$ also $CF\in\mathcal{N}$.

We now consider the parity symmetry. We define the parity transformation P, with $\mathcal{P}(u^0,\vec{u}) = (u^0, -\vec{u})$, by

$$\begin{aligned} P\psi_{\alpha,a}(u) = (\gamma_0)_{\alpha\beta}\,\psi_{\beta,a}(\mathcal{P}u)\,, \\ P\bar{\psi}_{\alpha,a}(u) = \bar{\psi}_{\beta,a}(\mathcal{P}u)\,(\gamma_0)_{\beta\alpha}\,, \\ Pf(\{g_{uv}\}) = f(\{g_{\mathcal{P}u\mathcal{P}v}\})\ ,\ P(AB) = P(A)P(B)\,, \end{aligned}$$

where A and B are Grassmann monomials, and extend it by linearity. $PS = S$ and $\langle PF\rangle_0 = \langle F\rangle_0$ which implies $\langle PF\rangle = \langle F\rangle$. As $\langle PF\Theta PF\rangle = 0$ if $\langle F\Theta F\rangle = 0$, P can be lifted to an operator acting in

$\mathcal{H}$ and P commutes with the Hamiltonian H. In particular, $P\bar{\phi}_-(u) = -\bar{\phi}_-(u^0, -\vec{u})$ and $P\phi_+(u) = \phi_+(u^0, -\vec{u})$, so that the improper states, with $u^0 = 1/2$, $\bar{\Phi}_-(\vec{p}) \equiv \sum_{\vec{u}\in\mathbb{Z}^2} e^{-i\vec{p}.\vec{u}}\bar{\phi}_-(u)$ and $\Phi_+(\vec{p}) \equiv \sum_{\vec{u}\in\mathbb{Z}^2} e^{-i\vec{p}.\vec{u}}\phi_+(u)$ satisfy $P\bar{\Phi}_-(\vec{0}) = -\bar{\Phi}_-(\vec{0})$ and $P\Phi_+(\vec{0}) = \Phi_+(\vec{0})$, i.e. $\bar{\Phi}_-(\vec{p} = \vec{0})$ and $\Phi_+(\vec{p} = \vec{0})$ have eigenvalues $+1$ and -1, respectively, and we identify these eigenvalues as the intrinsic parities.

Another consequence associated with the invariance of expectations under the parity transformation is that $w(\vec{p}) = w(-\vec{p})$; this follows since $\langle \phi_-(x)\bar{\phi}_-(y)\rangle = \langle P[\phi_-(x)\bar{\phi}_-(y)]\rangle = \langle \phi_-(x^0, -\vec{x})\bar{\phi}_-(y^0, -\vec{y})\rangle$ which implies $\tilde{G}(p^0, -\vec{p}) = \tilde{G}(p^0, \vec{p})$ and $\tilde{\Gamma}(p^0, -\vec{p}) = \tilde{\Gamma}(p^0, \vec{p})$. A spatial rotation by $\pi/2$ is also a symmetry and is defined by a transformation R similar to that of parity replacing P by R and $\mathcal{P}$ by $\mathcal{R}$, where $\mathcal{R}(x^0, x^1, x^2) = (x^0, x^2, -x^1)$ and γ_0 by either $(1 + \gamma_0)/2 + i(1 - \gamma_0)/2$, or with $-i$. With these choices, $R^2 = P$.

Next, we show how the interplay of the hyperplane decoupling method and gauge invariance not only reveals the lowest e-m excitations but also shows that the excitations are particles, i.e. have isolated dispersion curves. Consider the correlation function of two functions $H(x)$ and $L(y)$ localized at x and y, respectively, with $x^0 < y^0$. We assume that $\langle H(x)\rangle = 0 = \langle L(y)\rangle$. In the hyperplane decoupling method (see [6, 15]), the parameter κ in the action is replaced by the complex parameter κ_p for all bonds connecting the hyperplane $u^0 = p$ and $u^0 = p + 1$. We consider the resulting $\langle H(x)L(y)\rangle$ and its derivatives at $\kappa_p = 0$. Now, with $G(x, y, \kappa_p) \equiv \langle H(x)L(y)\rangle$, $G(x, y, \kappa_p = 0) = 0$ and if the first nonvanishing derivative, say the derivative of order k, is non-vanishing, and if we can find $L(x)$ and $H(y)$ such that, at $\kappa_p = 0$,

$$\frac{\partial^k}{\partial\kappa_p^k} G(x, y, 0) \propto \sum_{\vec{z}\in\mathbb{Z}^2} G(x, (p, \vec{z}))G((p + 1, \vec{z}), y)\,,$$

then, for the k-th derivative of minus the convolution inverse Γ' ($\Gamma' G = -1$), given by (Leibniz's rule) $\frac{\partial^k}{\partial\kappa_p^k}\Gamma'|_{\kappa_p=0} = \sum_{n=0}^{k-1}\begin{pmatrix} k \\ n \end{pmatrix}\Gamma' \frac{\partial^{k-n}G}{\partial\kappa_p^{k-n}}\frac{\partial^n\Gamma'}{\partial\kappa_p^n}|_{\kappa_p=0}$, only the $\ell = k$, $n = 0$ term contributes. Using the above structure of $\frac{\partial^k}{\partial\kappa_p^k} G\,|_{\kappa_p=0}$, we conclude that $\frac{\partial^k}{\partial\kappa_p^k}\Gamma'\,|_{\kappa_p=0} = 0$, $x^0 \le p < y^0 - 1$. This is the important ingredient that leads to the faster decay rate for Γ, as compared to G, i.e. $\mathrm{const}\, e^{-4|x^0 \ln\kappa|}$. The decay is obtained by repeating the argument for each hyperplane with $x^0 \le p \le y^0 - 1$, using joint analyticity in κ_p, $x^0 \le p < y^0$, and Cauchy estimates for the κ_p derivatives (see [6, 15]).

In our case, the coefficient of κ_p^n in the numerator of G is of the form

$$\frac{1}{n!}\int H(x)\left(\sum_{\vec{z}\in\mathbb{Z}^2}\dots\right)^n L(y)e^{-S(\bar{\psi},\psi,g)}\,d\psi d\bar{\psi} d\mu(g)\,|_{\kappa_p=0}\,.$$

We now perform the integral over the gauge field for the bonds between the hyperplanes. For $n = 1$, by the Peter-Weyl (see [16]) orthogonality relations the integration gives zero. For $n = 2$, if H and L are in the odd subspace of $\mathcal{H}$, then the Fermi integration gives zero. For $n = 3$, the gauge integral is zero unless the three bonds coincide. In the case of free Fermi fields, there is no such restriction. For coincident bonds a three-fold tensor product, where each factor is either U or $\bar{U}$ (with the bar denoting complex conjugate) occurs. Using the decomposition of tensor products of $SU(3)$, namely $3\times 3\times 3 = 10 + 8 + 8 + 1$, $3\times 3\times\bar{3} = 15 + \bar{6} + 3 + 3$, and their complex conjugates, only the identity representation contributes and the third derivative can be written, at $\kappa_p = 0$,

$$\frac{1}{3!}\frac{\partial^3\langle H(x)L(y)\rangle}{\partial\kappa_p^3} = -8\sum_{\vec{z}\in\mathbb{Z}^2}\big[\langle\bar{\phi}_+(p+1,\vec{z})L(y)\rangle \times \langle H(x)\phi_+(p,\vec{z})\rangle - \langle H(x)\bar{\phi}_-(p,\vec{z})\rangle\langle\phi_-(p+1,\vec{z})L(y)\rangle\big]\,.$$

Choosing $H(x) = \phi_-(x)$ and $L(y) = \bar{\phi}_-(y)$, we have, at $\kappa_p = 0$,

$$\frac{1}{3!}\frac{\partial^3 G_-}{\partial\kappa_p^3}(x, y, 0) = 8\sum_{\vec{z}\in\mathbb{Z}^2} G_-\left(x, (p,\vec{z})\right) G_-\left((p+1,\vec{z}), y\right),$$

such as, the required derivative structure. Similar considerations apply for G_+, choosing $H(x) = \bar{\phi}_+(x)$ and $L(y) = \phi_+(y)$, and $x^0 > p \ge y^0$.

We now show that the only e-m spectrum in all of $\mathcal{H}_o$, up to the energy value $-(4-\epsilon)\ln\kappa$, is the isolated baryon dispersion curve $w(\vec{p}) \equiv w_\pm(\vec{p})$. It is enough to show that for $L \in \mathcal{H}_o$ with finite spatial support and time support at $u^0 = 1/2$, the only possible e-m contribution, up to energies $-(4 - \epsilon)\ln\kappa$, in $(L, e^{-Hx^0}e^{i\vec{P}.\vec{x}}L)_{\mathcal{H}}$ comes from $w(\vec{p})$. By the F-K formula $(L, e^{-Hx^0}e^{i\vec{P}.\vec{x}}L)_{\mathcal{H}} = \langle L(x)\hat{L}\rangle$, $x^0 \ge 0$, where $\hat{L} \equiv \Theta L$, and in general we write $K(x)$ for the translation of K by x. Letting $G(x) = \langle L(x^0 - 1, \vec{x})\hat{L}\rangle$, $x \in \mathbb{Z}^3$, and using the F-K formula, the Fourier transform $\tilde{G}(p)$ of $G(x)$, has the spectral representation

$$\begin{aligned}\tilde{G}(p) = \tilde{G}(\vec{p}) &+ (2\pi)^{d-1}\textstyle\int_{-1}^{1}\int_{\mathbf{T}^2}\frac{1}{e^{ip^0}-\lambda^0}\delta(\vec{p}-\vec{\lambda})\\ &\times\ d_{\lambda^0}d_{\vec{\lambda}}(\hat{L},\mathcal{E}(\lambda^0)\mathcal{F}(\vec{\lambda})\hat{L})_{\mathcal{H}} - (2\pi)^{d-1}\textstyle\int_{-1}^{1}\int_{\mathbf{T}^2}\\ &\times\ \frac{1}{e^{-ip^0}-\lambda^0}\delta(\vec{p}+\vec{\lambda})\,d_{\lambda^0}d_{\vec{\lambda}}(L,\mathcal{E}(\lambda^0)\mathcal{F}(\vec{\lambda})L)_{\mathcal{H}}\,,\end{aligned} \tag{3}$$

where $\tilde{G}(\vec{p}) = \sum_{\vec{x}\in\mathbb{Z}^2} e^{-i\vec{p}.\vec{x}}G(x^0 = 0, \vec{x})$. We define, with $G(x, y) = G(x - y)$, $F(x, y) = G(x, y) + \sum_{u,v\in\mathbb{Z}^3}\langle L(x)\Phi(u)\rangle\Gamma(u, v)\langle\chi(v)\hat{L}(y)\rangle$ where $\Phi(u) = (\phi_+(u), \bar{\phi}_-(u))$ and $\chi(v) = (\bar{\phi}_+(v), -\phi_-(v))$ have two components; $\Gamma(u, v)$ is a 2×2 matrix and we suppress the sum over components, Γ is minus the convolution inverse of M, where $M(x, y) = \mathrm{diag}\left[\langle\bar{\phi}_+(x)\phi_+(y)\rangle\,;\,-\langle\phi_-(x)\bar{\phi}_-(y)\rangle\right]$. The second term in $F(x, y)$ is designed so that we have: *i)* that the possible singularities, in $|\mathcal{I}m\ p^0| < -(4 - \epsilon)\ln\kappa$, in $\tilde{G}(p)$ are cancelled; *ii)* the only possible singularities in the Fourier transform come from $w(\vec{p})$.

By the hyperplane decoupling method applied to $F(x,y)$, $x^0 \leq p < y^0$, it is seen that $\partial^k F(x,y)/\partial\kappa_p^k = 0$, for $k = 0,1,2,3$, so that $|F(x,y)| \leq$ const $e^{-(4-\epsilon)|\ln(\kappa)\,(x^0-y^0)|}$, which implies analyticity of $\tilde{F}(p)$, the Fourier transform of $F(x,0)$, in $|\mathcal{I}\mathrm{m}\ p^0| < -(4-\epsilon)\ln\kappa$. For $\tilde{F}(p)$, we have $\tilde{F}(p) = \tilde{G}(p) + \tilde{H}(p)\tilde{\Gamma}(p)\tilde{J}(p)$, where $\tilde{H}(p)$, $\tilde{\Gamma}(p)$ and $\tilde{J}(p)$ are the Fourier transforms of $\langle L(x)\Phi(0)\rangle$, $\Gamma(x,0)$ and $\langle\chi(x)\hat{L}(0)\rangle$, respectively. $\tilde{H}(p)$ and $\tilde{J}(p)$ have spectral representations which are similar to the one for $\tilde{G}(p)$ given in Eq. (3), with the only possible singularities given by $w(\vec{p})$ in $|\mathcal{I}\mathrm{m}\ p^0| < -(4-\epsilon)\ln\kappa$ and $\tilde{\Gamma}(p)$ is analytic in this region. It follows that the only possible singularities of $\tilde{G}(p)$ are those arising from $w(\vec{p})$, thus completing our argument.

A complete and more detailed account of our results will appear elsewhere.

Acknowledgments

This work was partially supported by Pronex, CNPq and FAPESP.

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