Spectral Properties of Weakly Coupled Landau-Ginzburg Stochastic Models

Ricardo Schor[1], João C. A. Barata[2], Paulo A. Faria da Veiga[3], Emmanuel Pereira[1]

[1]Dep. Física-ICEx, UFMG, CP 702, 30.161-970 Belo Horizonte MG, Brazil

[2]IFUSP-USP, CP 66318, 05315-970 São Paulo SP, Brazil

[3]ICMC-USP, CP 668, 13560-970 São Carlos SP, Brazil

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Abstract

We study existence of bound states in the generator of the stochastic dynamics associated to weakly coupled lattice Landau-Ginzburg models. By analyzing the Bethe-Salpeter kernel in the ladder approximation, these states are shown to exist if the polynomial interaction has a negative quartic term and the lattice dimension is smaller than three. Asymptotic values for the masses are also obtained, giving precise relaxation rates for even correlations.

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1 Introduction

In this paper, we consider some aspects for the stochastic dynamics of lattice systems described by an action of the form

$$S(\varphi) = \sum_{\vec{x}\in\mathbb{Z}^d} \left\{ \frac{1}{2}\left[\sum_{i=1}^{d}(\varphi(\vec{x}+\vec{e}_i)-\varphi(\vec{x}))^2 + m^2\varphi(\vec{x})^2\right] + \lambda\mathcal{P}(\varphi(\vec{x}))\right\} \tag{1}$$

where $\varphi(\vec{x})$ is a real continuous spin variable at $\vec{x}\in\mathbb{Z}^d$, the unit d-dimensional lattice, $\vec{e}_i$ is the unit vector along the ith coordinate, $\mathcal{P}$ is an even polynomial bounded from below, $m>0$ and

$\lambda \geq 0$.

For $t \in \mathbb{R}$ denoting the time variable, the dynamics is introduced by the Langevin equation

$$\frac{\partial}{\partial t}\varphi(\vec{x},t) = -\frac{1}{2}\frac{\delta}{\delta\varphi(\vec{x},t)}S + \eta(\vec{x},t) \tag{2}$$

where $\{\eta(\vec{x},t)\}$ is a family of Gaussian white noise processes with the expectations $E(\eta(\vec{x},t)) = 0$ and $E(\eta(\vec{x},t)\eta(\vec{x}',t')) = \delta_{\vec{x},\vec{x}'}\delta(t-t')$. Such models can be used to describe the (purely relaxational) evolution of an order parameter in statistical mechanical systems [1,2].

The dynamics induced by (2) is associated with a Markov semigroup and leaves invariant the Gibbs distribution $d\mu = e^{-S(\varphi)}d\varphi/normalization$ defined by the action (1). More specifically, if f is any function of the spin configuration $\varphi = \{\varphi(\vec{x})\}$, we define its time evolution f_t by

$$f_t(\psi) = E(f(\varphi(t))) \tag{3}$$

where $\varphi(0) = \psi$ is the initial condition in eq.(2). It follows then that f_t is determined by the Markov semigroup e^{-tH} whose generator H, given by

$$Hf = -\frac{1}{2}\sum_{\vec{x}\in\mathbb{Z}^d}\left[\frac{\partial^2}{\partial\varphi(\vec{x})^2}f - \frac{\partial S}{\partial\varphi(\vec{x})}\frac{\partial f}{\partial\varphi(\vec{x})}\right] \tag{4}$$

is positive and Hermitian on $L^2(d\mu)$. Clearly, the constant function $f = 1$ is an eigenfunction of H with zero eigenvalue. As m is nonzero, for small λ, there is a gap in the spectrum of H, implying an exponentially fast approach to equilibrium in the system.

It is possible and indeed desirable to associate a quasiparticle structure to the operator H, which can then be viewed as a Hamiltonian, since this structure provides information on corrections to the exponential law of approach to equilibrium. Momentum operators $\vec{P}$ are naturally defined by space translations and commute with H. ¿From this point of view, the natural question to ask is about the nature of the spectrum of $(H, \vec{P})$.

This problem has been recently considered by Kondratiev and Minlos [3], in the context of the stochastic XY model at high temperatures. They constructed one-particle states (for two different species of quasiparticles) and showed that they are isolated from the rest of the spectrum.

The existence of isolated one-particle states for the model defined by (1) and (2), which is assumed in this work, follows by adapting standard techniques of constructive field theory [4], [7], developed to study analyticity properties of one-particle irreducible Green's functions, taking as input the convergent cluster expansion established by Dimock in ref. [5]. This paper intends to further our knowledge about the spectrum of $(H, \vec{P})$. More precisely, we analyze the existence of bound states of two quasi-particles. We remark that the mass of such a bound state shows up directly in the exponential approach to equilibrium for even observables. Hence this question is of direct physical relevance.

To attack this problem, we use a functional integral representation for the associated correlation functions and look at the dynamical system as a quantum field theory in discrete coordinate space and continuous time. This field theory turns out to present non-local interactions. The part of the spectrum above the one-particle states is studied through a Bethe-Salpeter (B-S) equation in a way which is similar to the methods employed previously in local relativistic field theory. Here, however, the discreteness of space and the non-locality of the interactions represent additional complications in the analysis of the B-S kernel. In order to simplify them, we do restrict ourselves to the spectral analysis of translationally invariant states. Also, the B-S kernel is computed only in the ladder approximation. This procedure has been justified in cases where a rigorous analysis was possible [6].

To state our results, we write the polynomial interaction $\mathcal{P}$ in (1) as an expansion in an Hermite basis (see Section 3), starting with the fourth power:

$$\mathcal{P}(x) = \sum_{n=2}^{N} \frac{a_n}{(2n)!} : x^{2n} : \tag{5}$$

with $a_N > 0$. If λ is small, we show (in the ladder approximation) absence of bound states for spatial dimension $d \geq 3$, as well as for $d < 3$ and $a_2 \geq 0$. For $d < 3$ and $a_2 < 0$, there is a unique bound state. In dimension one, the mass of this bound state is

$$M^* = 2M - \frac{9a_2^2\lambda^2}{4m^4}(1 + \mathcal{O}(\lambda)) \tag{6}$$

and, for $d = 2$,

$$M^* = 2M - \exp\left[-\frac{4\pi m^2}{3|a_2|\lambda}(1 + \mathcal{O}(\lambda))\right] \tag{7}$$

Above, M is the mass of the single quasi-particle.

This paper is organized as follows. In Section 2, we discuss on the functional integration representation and the form of the B-S equation which is suitable to handle field theories on a lattice. The computation of the B-S kernel and the mass spectrum above the one-particle state are presented in Section 3. Section 4 is devoted for conclusions.

2 The Feynman-Kac Formula

Consider the Hamiltonian (4) on a finite hypercube $\Lambda \subset \mathbb{Z}^d$ with periodic boundary conditions:

$$H_\Lambda f = -\frac{1}{2}\sum_{\vec{x}\in\Lambda}\left[\frac{\partial^2}{\partial\varphi(\vec{x})^2}f - \frac{\partial S}{\partial\varphi(\vec{x})}\frac{\partial f}{\partial\varphi(\vec{x})}\right] \tag{8}$$

For $d\varphi_\Lambda = \prod_{\vec{x}\in\Lambda} d\varphi(\vec{x})$, let

$$d\mu_\Lambda(\varphi) = \frac{1}{Z_\Lambda}e^{-S_\Lambda}d\varphi_\Lambda \tag{9}$$

with S_Λ given by (1), restricted to Λ with periodic boundary conditions, and where Z_Λ is a normalization for $d\mu_\Lambda$ so that $\int d\mu_\Lambda = 1$. With this, the operator H_Λ is Hermitian on the space $L^2(d\mu_\Lambda)$. Next, let U_Λ be the unitary operator from $L^2(d\mu_\Lambda)$ to $L^2(d\varphi_\Lambda)$ given by

$$(U_\Lambda f)(\varphi) = Z_\Lambda^{-1/2}e^{-\frac{1}{2}S_\Lambda}f(\varphi) \tag{10}$$

A straightforward calculation shows that

$$L_\Lambda = U_\Lambda H_\Lambda U_\Lambda^{-1} = -\frac{1}{2}\sum_{\vec{x}\in\Lambda}\frac{\partial^2}{\partial\varphi(\vec{x})^2} + \frac{1}{4}\sum_{\vec{x}\in\Lambda}\left[\frac{1}{2}\left(\frac{\partial S_\Lambda}{\partial\varphi(\vec{x})}\right)^2 - \frac{\partial^2 S_\Lambda}{\partial\varphi(\vec{x})^2}\right] \tag{11}$$

so that L_Λ is a Schrödinger type operator. Performing the derivatives, we get

$$\begin{aligned} L_\Lambda = & -\frac{1}{2}\sum_{\vec{x}\in\Lambda}\frac{\partial^2}{\partial\varphi(\vec{x})^2} + \frac{1}{8}\sum_{\vec{x}\in\Lambda}\varphi(\vec{x})[(-\Delta + m^2)^2\varphi](\vec{x}) \\ & + \frac{\lambda}{4}\sum_{\vec{x}\in\Lambda}[(-\Delta + m^2)\varphi](\vec{x})\mathcal{P}'(\varphi(\vec{x})) + \sum_{\vec{x}\in\Lambda}\left[\frac{\lambda^2}{8}\mathcal{P}'(\varphi(\vec{x}))^2 - \frac{\lambda}{4}\mathcal{P}''(\varphi(\vec{x})) - \frac{(2d + m^2)}{4}\right] \end{aligned} \tag{12}$$

In the above formula, $-\Delta$ is the lattice Laplacian with periodic boundary conditions on Λ given by

$$(-\Delta\varphi)(\vec{x}) = 2d\varphi(\vec{x}) - \sum_{|\vec{x}-\vec{y}|=1} \varphi(\vec{y}) \tag{13}$$

The functional integral associated with (12) can be obtained by standard methods [4]. If $f_1, \ldots, f_n$ are functions of the spin configuration in Λ, if $\Omega(\varphi) = 1$ is the ground state of H_Λ and for $t_1 \leq t_2 \leq \ldots t_n \in \mathbb{R}$, then we have the Feynman-Kac formula

$$\begin{aligned}
&(\Omega, f_1 e^{-(t_2-t_1)H_\Lambda} f_2 \ldots e^{-(t_n-t_{n-1})H_\Lambda} f_n \Omega)_{L^2(d\mu_\Lambda)} \\
&= (U_\Lambda\Omega, f_1 e^{-(t_2-t_1)L_\Lambda} f_2 \ldots e^{-(t_n-t_{n-1})L_\Lambda} f_n U_\Lambda \Omega)_{L^2(d\varphi_\Lambda)} \\
&= \int f_1(\varphi(t_1)) \ldots f_n(\varphi(t_n)) d\rho_\Lambda
\end{aligned} \tag{14}$$

where the path space measure $d\rho_\Lambda$ is given by

$$d\rho_\Lambda = \frac{e^{-W_\Lambda} d\nu_\Lambda}{\int e^{-W_\Lambda} d\nu_\Lambda} \tag{15}$$

with

$$W_\Lambda = \int_{-\infty}^{\infty} dt \sum_{\vec{x}\in\Lambda} \left[\frac{\lambda}{4}\mathcal{P}'(\varphi(\vec{x},t))(-\Delta + m^2)\varphi(\vec{x},t) + \frac{\lambda^2}{8}\mathcal{P}'(\varphi(\vec{x},t))^2 - \frac{\lambda}{4}\mathcal{P}''(\varphi(\vec{x},t)) \right] \tag{16}$$

and $d\nu_\Lambda$ is a Gaussian measure with mean zero and variance given by

$$\int \varphi(\vec{x},t)\varphi(\vec{y},t')d\nu_\Lambda = \frac{1}{2\pi|\Lambda|} \int_{-\infty}^{\infty} dp_0 \sum_{\vec{p}\in\tilde{\Lambda}} \frac{e^{ip_0(t-t')}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{p_0^2 + \left[\sum_{i=1}^d (1-\cos p_i) + \frac{m^2}{2}\right]^2} \tag{17}$$

In (17), $|\Lambda|$ is the number of points in Λ, $\tilde{\Lambda}$ is the Fourier dual lattice, $\vec{p} = (p_1, \ldots, p_d) \in \tilde{\Lambda}$ and $\vec{p}\cdot(\vec{x}-\vec{y}) = \sum_{i=1}^d p_i(x_i - y_i)$.

The thermodynamic limit $\Lambda \to \mathbb{Z}^d$ can now be taken in equation (14). The corresponding limiting expressions for (12-16) are easily obtained. The normalized sum $\frac{1}{|\Lambda|}\sum_{\vec{p}\in\tilde{\Lambda}}$ in the propagator (17) is replaced in the limit by an integral $\frac{1}{(2\pi)^d}\int_{T_d} d^d p$ over the d-dimensional torus $T_d = [-\pi,\pi]^d$. That the limit $\Lambda \to \mathbb{Z}^d$ exists, at least for small λ, follows from a cluster expansion argument (see [5]). Dropping hereafter the subscript Λ for the infinite-volume quantities, we have the representation

$$(\Omega, \hat{\varphi}(\vec{x}_1) e^{-(t_2-t_1)H} \hat{\varphi}(\vec{x}_2) \dots e^{-(t_n-t_{n-1})H} \hat{\varphi}(\vec{x}_n)\Omega) = \\ = \int \varphi(\vec{x}_1, t_1) \dots \varphi(\vec{x}_n, t_n) d\rho \equiv S_\lambda^{(n)}(\vec{x}_1, t_1; \dots; \vec{x}_n, t_n) \tag{18}$$

where the inner product $(\cdot, \cdot)$ on the lhs is taken on the physical Hilbert space $L^2(d\mu)$, $\hat{\varphi}(x)$ is the zero time field at $\vec{x} \in \mathbb{Z}^d$, and $t_1 \leq t_2 \leq \cdots t_n \in \mathbb{R}$. Since the infinite volume theory is translational invariant, we can introduce momentum operators $\vec{P}$, commuting with H, such that (writing $\hat{\varphi}(0) = \hat{\varphi}$)

$$(\Omega, \hat{\varphi}(\vec{x}_1) e^{-(t_2-t_1)H} \hat{\varphi}(\vec{x}_2) \dots e^{-(t_n-t_{n-1})H} \hat{\varphi}(\vec{x}_n)\Omega) \\ = (\Omega, \hat{\varphi} e^{-(t_2-t_1)H + i\vec{P}\cdot(\vec{x}_2-\vec{x}_1)} \hat{\varphi} \dots e^{-(t_n-t_{n-1})H + i\vec{P}\cdot(\vec{x}_n-\vec{x}_{n-1})} \hat{\varphi}\Omega) \tag{19}$$

For $p = (p_0, \vec{p})$, $p_0 \in \mathbb{R}$ and $\vec{p} \in T_d$, let

$$\tilde{S}_\lambda^{(2)}(p) = \int_{-\infty}^{\infty} dt \sum_{\vec{x} \in \mathbb{Z}^d} S_\lambda^{(2)}(\vec{0}, 0; \vec{x}, t) e^{-i(p_0 t + \vec{p}\cdot\vec{x})} \tag{20}$$

It follows from (18), (19) and the spectral theorem that

$$\tilde{S}_\lambda^{(2)}(p) = \int_0^\infty \int_{T_d} \frac{2E}{E^2 + p_0^2} (2\pi)^d \delta(\vec{q} - \vec{p}) d(\Omega, \hat{\varphi}\mathcal{E}(E, \vec{q})\hat{\varphi}\Omega) \tag{21}$$

where $\mathcal{E}(E, \vec{p})$ is the spectral projection associated with the operators $(H, \vec{P})$. The integral over E runs from 0 to ∞ and that over $\vec{q}$ is on T_d. We can write (21) in the form

$$\tilde{S}_\lambda^{(2)}(p) = \int_0^\infty \frac{2E}{E^2 + p_0^2} d\eta_\lambda(E; \vec{p}) \tag{22}$$

where the positive measure $d\eta_\lambda(E, \vec{p})$ is supported on the spectrum of H restricted to the odd states with momentum $\vec{p}$. When $\lambda = 0$, we have from (17)

$$\tilde{S}_\lambda^{(2)}(p) = \frac{1}{p_0^2 + E_0(\vec{p})^2} \quad ; \quad E_0(\vec{p}) = \sum_{i=1}^d (1 - \cos p_i) + \frac{1}{2} m^2 \tag{23}$$

$E_0(\vec{p})$ is identified with the energy of an elementary excitation (quasiparticle) with momentum $\vec{p}$ and mass $E_0(\vec{0}) = M_0 = m^2/2$ in the free, i.e. the $\lambda = 0$, case.

When λ is small, $\tilde{S}_\lambda^{(2)}(p)$ has the representation (see Section 3)

$$\tilde{S}_\lambda^{(2)}(p) = \frac{c_\lambda(\vec{p})}{p_0^2 + E_\lambda(\vec{p})^2} + \int_{2M_0}^{\infty} \frac{2E}{E^2 + p_0^2} d\eta_\lambda(E;\vec{p}) \tag{24}$$

$E_\lambda(\vec{p})$ is the dispersion function in the interacting theory and, as will be shown in the next section, it differs from $E_0(\vec{p})$ by $\mathcal{O}(\lambda^2)$. Thus, if m is large, $E_\lambda(\vec{p})$ is isolated from the rest of the spectrum. The mass of the interacting quasiparticle is $M_\lambda = E_\lambda(\vec{0}) = M_0 + \mathcal{O}(\lambda^2)$.

To study the spectrum of H on even states, consider the truncated four-point function

$$D_\lambda(x_1, x_2; x_3, x_4) = S_\lambda^{(4)}(x_1, x_2, x_3, x_4) - S_\lambda^{(2)}(x_1, x_2) S_\lambda^{(2)}(x_3, x_4) \tag{25}$$

where $x_i = (t_i, \vec{x}_i)$. From translation invariance, D_λ depends only on difference variables. Let $\xi = x_2 - x_1$, $\eta = x_4 - x_3$ and $\tau = x_3 - x_2$. Writing $\xi = (\xi_0, \vec{\xi})$, etc, it follows from (19) that if $\xi_0 = \eta_0 = 0$,

$$D_\lambda(\xi, \eta, \tau) = (\theta(-\vec{\xi}), e^{-|\tau|H} e^{i\vec{P}\cdot\vec{\tau}} \theta(\vec{\eta})) \tag{26}$$

where

$$\theta(\vec{\eta}) = \hat{\varphi}(\vec{0})\hat{\varphi}(\vec{\eta})\Omega - (\Omega, \hat{\varphi}(\vec{0})\hat{\varphi}(\vec{\eta})\Omega)\Omega \tag{27}$$

Let $f : \mathbb{Z}^d \to \mathbb{C}$ be an arbitrary function vanishing outside a finite set and let $\tilde{f}(\vec{p})$ and $\tilde{D}_\lambda(p, q, k)$ be respectively the Fourier transforms of $f(\vec{x})$ and $D_\lambda(\xi, \eta, \tau)$, defined as in (20). A simple calculation shows that

$$\begin{aligned} &\int \int_{-\infty}^{\infty} \int \int_{T_d} d^{d+1}p \;\; d^{d+1}q \;\; \overline{\tilde{f}}(\vec{p}) \tilde{f}(\vec{q}) \tilde{D}_\lambda(p, q, k) = \\ &\quad = \int_0^\infty \int_{T_d} \frac{2E}{k_0^2 + E^2} (2\pi)^{3d+2} \delta(\vec{q} - \vec{k}) d(\theta(f), \mathcal{E}(E, \vec{q})\theta(f)) \end{aligned} \tag{28}$$

where

$$\theta(f) = \sum_{\vec{x} \in \mathbb{Z}^d} f(\vec{x}) \theta(-\vec{x}) \tag{29}$$

Formula (28) is similar to (21). The singularities in k_0, for fixed $\vec{k}$, of the lhs of (28) give direct information about the spectrum of H on the even subspace of states with momentum $\vec{k}$.

To test for the presence of bound states, it is sufficient to study the spectrum on the subspace of zero-momentum states.

When $\lambda = 0$ and $\vec{k} = 0$, a direct calculation shows that

$$(\tilde{f}, \tilde{D}_0(k_0, \vec{k} = 0), \tilde{f})_{L^2} = \frac{\pi}{4}(2\pi)^{d+1} \int_{T_d} d^d p \frac{|\tilde{f}(\vec{p}) + \tilde{f}(-\vec{p})|^2}{E_0(\vec{p})[E_0(\vec{p})^2 + \frac{1}{4}k_0^2]} \tag{30}$$

where the lhs of (30) is a short notation for the lhs of (28). The rhs in (30) is analytic in k_0 for $|Imk_0| < 2M_0$. Then, for the free ($\lambda = 0$) theory, there is no energy spectrum in the interval $(0, 2M_0)$ for even states with zero momentum, as it should be.

We will show in the next Section that, if $\lambda > 0$ and the (normal ordered) interacting polynomial has a negative quartic term, then the left hand side of (28) has a singularity on the positive imaginary axis below $2M_\lambda$ if $d \leq 2$. Therefore, in this case, we do get two-particle bound states.

3 Analysis of Bound States

The analysis of bound states in local relativistic field theories using the Bethe-Salpeter equation is well known. In our particular problem, we deal with a slightly non-local field theory on a lattice space. The non-locality makes the Bethe-Salpeter kernel more complicated and the lattice makes unsuitable the use of canonical relative coordinates, as in e.g. [6]. Nevertheless, using the coordinates ξ, η and τ, defined before (26) most of the analysis can be done in the standard way. Therefore we will be brief.

For ease of computation, the interacting polynomial (5) is written as an Hermite expansion, with the generating function for the monomials $: x^k :$ given by

$$: e^{i\alpha x} := e^{i\alpha x} e^{-\frac{1}{2}\alpha^2 c} \tag{31}$$

where $\alpha \in \mathbb{R}$ and

$$c = \frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dp_0 \int_{T_d} d^d p \frac{1}{p_0^2 + E_0(\vec{p})^2} = S_0^{(2)}(\vec{0}, 0; \vec{0}, 0) \tag{32}$$

Let $\tilde{\Gamma}_\lambda(p) = \tilde{S}^{(2)}_\lambda(p)^{-1}$ and let $\tilde{k}_\lambda(p) = \tilde{\Gamma}_0(p) - \tilde{\Gamma}_\lambda(p)$, so that Dyson's equation

$$\tilde{S}^{(2)}_\lambda = \tilde{S}^{(2)}_0 + \tilde{S}^{(2)}_0 \tilde{k}_\lambda \tilde{S}^{(2)}_\lambda \tag{33}$$

is satisfied.

Diagrammatically, $\tilde{k}_\lambda$ is the sum of all one-particle-irreducible Feynman graphs with two external legs. This implies $\tilde{k}_\lambda$ and hence $\tilde{\Gamma}_\lambda$ to be analytic on $|\mathrm{Im} p_0| < 2M_0$ for real $\vec{p}$. Also, since $\tilde{k}_\lambda$ is $\mathcal{O}(\lambda)$ near the zeroes $(\pm iE_0(\vec{p}), \vec{p})$ of $\tilde{\Gamma}_0(p)$, it follows that $\tilde{\Gamma}_\lambda(p)$ has zeroes nearby, which we call $(\pm iE_\lambda(\vec{p}), \vec{p})$ with $E_\lambda(\vec{p}) - E_0(\vec{p}) = \mathcal{O}(\lambda)$. That these zeroes can only be located on the imaginary axis follows from general principles, see eq. (21). The representation (24) with $c_\lambda(\vec{p}) = 1 + \mathcal{O}(\lambda)$ follows immediately from the above facts. Actually, we have $E_\lambda(\vec{p}) - E_0(\vec{p}) = \mathcal{O}(\lambda^2)$ since $\frac{\partial}{\partial\lambda}\tilde{\Gamma}_\lambda(p)|_{\lambda=0} = 0$ by explicit computation.

We next study the truncated four point function (25) using the Bethe-Salpeter equation

$$D_\lambda = D^0_\lambda + D^0_\lambda K_\lambda D_\lambda \tag{34}$$

where D_λ, etc, are operators defined by the kernels $D_\lambda(x_1, x_2; x_3, x_4)$, etc, and

$$D^0_\lambda(x_1, x_2; x_3, x_4) = S^{(2)}_\lambda(x_1, x_3) S^{(2)}_\lambda(x_2, x_4) + S^{(2)}_\lambda(x_1, x_4) S^{(2)}_\lambda(x_2, x_3) \tag{35}$$

The Bethe-Sapeter kernel $K_\lambda(x_1, x_2; x_3, x_4)$ is the sum of all connected Feynman diagrams with four (amputated) external lines which are (channel) two-particle irreducible. Introducing the relative coordinates ξ, η and τ as in Section 2, it follows that the Fourier transform of the kernels of D_λ, D^0_λ and K_λ satisfy an equation similar to (34):

$$\tilde{D}_\lambda(k) = \tilde{D}^0_\lambda(k) + (2\pi)^{-2(d+1)} \tilde{D}^0_\lambda(k) \tilde{K}_\lambda(k) \tilde{D}_\lambda(k) \tag{36}$$

where e.g. $\tilde{D}_\lambda(k)$ is defined by the kernel $\tilde{D}_\lambda(p, q, k)$, i.e.,

$$(\tilde{D}_\lambda(k) f)(p) = \int_{-\infty}^{\infty} dq_0 \int_{T_d} d^d q \tilde{D}_\lambda(p, q, k) f(q) \tag{37}$$

The ladder approximation which we adopt here consists of replacing $\tilde{K}_\lambda$ by its first order term $\tilde{L}_\lambda$ in the perturbation expansion. Explicit calculation shows that

$$\tilde{L}_\lambda(p, q, k) = -\frac{3}{4} a_2 \lambda [E_0(\vec{p}) + E_0(\vec{q}) + E_0(\vec{p} - \vec{k}) + E_0(\vec{q} - \vec{k})] \tag{38}$$

At zero total momentum,

$$\tilde{L}_\lambda(p, q, (k^0, \vec{0})) = -\frac{3}{2} a_2 \lambda [E_0(\vec{p}) + E_0(\vec{q})] \tag{39}$$

We see that $\tilde{L}_\lambda(k^0, \vec{0})$ is a rank two operator, in contrast with what happens in a genuine local field theory, where the rank is just one.

Equation (36) with $\tilde{K}_\lambda$ replaced by $\tilde{L}_\lambda$ can be solved for $\tilde{D}_\lambda$ to yield

$$\begin{aligned} \tilde{D}_\lambda(k^0) &= [1 - (2\pi)^{-2(d+1)} \tilde{D}^0_\lambda(k^0) \tilde{L}_\lambda(k^0)]^{-1} \tilde{D}^0_\lambda(k^0) \\ &= \tilde{D}^0_\lambda(k^0) [1 - (2\pi)^{-2(d+1)} \tilde{L}_\lambda(k^0) \tilde{D}^0_\lambda(k^0)]^{-1} \end{aligned} \tag{40}$$

where $\tilde{D}_\lambda(k^0) = \tilde{D}_\lambda((k^0, \vec{0}))$, etc.

¿From (35), one can show that the action of $\tilde{D}^0_\lambda(k^0)$ on functions $f(p)$ depending only on $\vec{p}$ is

$$(\tilde{D}^0_\lambda(k^0) f)(p) = (2\pi)^{d+1} \tilde{S}^{(2)}_\lambda(p) \tilde{S}^{(2)}_\lambda(k-p) [f(\vec{p}) + f(-\vec{p})] \tag{41}$$

Therefore, if f depends only on $\vec{p}$, we have

$$(\tilde{L}_\lambda(k^0) \tilde{D}_\lambda(k^0) f)(p) = -3 a_2 \lambda (2\pi)^{d+1} [\rho_0(f) + \rho_1(f) E_0(\vec{p})] \tag{42}$$

where

$$\rho_n(f) = \frac{1}{2} \int_{T_d} d^d q \;\; G(\vec{q}, k^0) E_0(\vec{q})^n [f(\vec{q}) + f(-\vec{q})] \quad ; \quad n = 0, 1 \tag{43}$$

and

$$G(\vec{q}, k^0) = \int_{-\infty}^{\infty} dq_0 \;\; \tilde{S}^{(2)}_\lambda(q) \tilde{S}^{(2)}_\lambda(k^0 - q_0, \vec{q}) \tag{44}$$

It follows from (24) and from a simple analytic continuation argument that $G(\vec{q}, k^0)$ is analytic on $|\mathrm{Im} k^0| < 2E_\lambda(\vec{0})$. This result depends on the fact that $E_\lambda(\vec{0}) \le E_\lambda(\vec{p})$ for any $\vec{p} \in T_d$, which holds because $S^{(2)}_\lambda(x, y) > 0$.

Recall, from (28), that the basic object we want to analyze is $(f, \tilde{D}_\lambda(k^0) f)$, which has the form

$$(f, \tilde{D}_\lambda(k^0) f) = 2(2\pi)^{d+1} \int_{T_d} d^d p \;\; \overline{f}(\vec{p}) G(\vec{p}, k^0) g(\vec{p}, k^0) \tag{45}$$

where

$$g(\cdot, k^0) = [1 - (2\pi)^{-2(d+1)}\tilde{L}_\lambda(k^0)\tilde{D}^0_\lambda(k^0)]^{-1}f \tag{46}$$

The only singularities of (45) on $|\mathrm{Im}k^0| < 2E_\lambda(\vec{0})$ must come from those of $g(\cdot, k^0)$, which in turn come from the zeroes of $1-\mu_\pm(k^0)$, where $\mu_\pm(k^0)$ are the eigenvalues of $(2\pi)^{-2(d+1)}\tilde{L}_\lambda(k^0)\tilde{D}^0_\lambda(k^0)$ on the space generated by the functions 1 and $E_0(\vec{p})$.

We find

$$\mu_\pm(k^0) = -3a_2\lambda(2\pi)^{-(d+1)}(\alpha(k^0) \pm [\beta(k^0)\gamma(k^0)]^{1/2}) \tag{47}$$

where

$$\alpha(k^0) = \int_{T_d} E_0(\vec{q})G(\vec{q}, k^0)d^dq, \quad \beta(k^0) = \int_{T_d} G(\vec{q}, k^0)d^dq, \quad \gamma(k^0) = \int_{T_d} E_0(\vec{q})^2 G(\vec{q}, k^0)d^dq \tag{48}$$

Now, from (24), $G(\vec{q}, k^0)$ can be written as

$$G(\vec{q}, k^0) = \frac{\pi}{2}\frac{c_\lambda(\vec{q})^2}{E_\lambda(\vec{q})[E_\lambda(\vec{q})^2 + \frac{1}{4}(k^0)^2]} + G_1(\vec{q}, k^0) \tag{49}$$

where $G_1(\vec{q}, k^0)$ is analytic on $|\mathrm{Im}k^0| < E_\lambda(\vec{0}) + 2M_0$.

¿From general principles, the singularities of (45) can only be located on the imaginary k^0 axis. Writing $k^0 = i\chi$ with $\chi \geq 0$, it is possible to show (using an explicit formula) that $G(\vec{q}, i\chi) > 0$ for $0 \leq \chi < 2E_\lambda(\vec{0})$. It follows then that $\alpha(i\chi)$, $\beta(i\chi)$ and $\gamma(i\chi)$ are positive and, by Schwarz's inequality, $\alpha \leq [\beta\gamma]^{1/2}$ on $0 \leq \chi < 2E_\lambda(\vec{0})$.

For space dimension $d \geq 3$, then $\alpha(i\chi)$, $\beta(i\chi)$ and $\gamma(i\chi)$ increase to a finite limit as $\chi \to 2E_\lambda(\vec{0})$ because the singularity generated by $G(\vec{q}, i\chi)$ is quadratic and therefore integrable. Thus, if λ is small enough, $1 - \mu_\pm(i\chi)$ cannot be zero on $0 < \chi < 2E_\lambda(\vec{0})$ so that, in the ladder approximation, there are no bound states.

If $d < 3$, α, β and γ diverge as $\chi \to 2E_\lambda(\vec{0})$, but $\alpha - [\beta\gamma]^{1/2}$ remains finite. This yields the nonvanishing of $1 - \mu_-(i\chi)$. Finally, $1 - \mu_+(i\chi)$ is nonzero if $a_2 > 0$, and has a unique zero on the interval $0 < \chi < 2E_\lambda(\vec{0})$, if $a_2 < 0$. This implies the existence of one bound state for the last case.

Let $M_\lambda = E_\lambda(\vec{0})$ be the mass for a single quasiparticle in the interacting theory. The mass M^* of the bound state is the solution of (assuming $a_2 < 0$)

$$F(\lambda, iM^*) = \frac{-(2\pi)^{d+1}}{3a_2\lambda} \tag{50}$$

where $F(\lambda, k^0) = \alpha(\lambda, k^0) + [\beta(\lambda, k^0)\gamma(\lambda, k^0)]^{1/2}$, and we have made explicit the λ dependence of α, β and γ. Let $\mathcal{E} = 2M_\lambda - M^*$. Performing an asymptotic analysis of the coefficients α, β, and γ we find

$$\mathcal{E}(\lambda) = \begin{cases} \frac{9}{4}\frac{\lambda^2}{m^4}a_2^2(1+\mathcal{O}(\lambda)) & ; \text{ if } \ \mathrm{d}=1 \\ \exp\left[-\frac{4\pi m^2}{3|a_2|\lambda}(1+\mathcal{O}(\lambda))\right] & ; \text{ if } \ \mathrm{d}=2 \end{cases} \tag{51}$$

4 Conclusions

In this paper, we have analyzed the existence of bound states for the generator of stochastic dynamics in purely relaxational lattice Landau-Ginzburg models. This problem is directly related to decay rates of some observables.

We have shown the existence of a bound state for small coupling if the polynomial interaction has a negative quartic term and the space dimension is one or two. This result was obtained by analyzing the Bethe-Salpeter kernel of the non-local lattice quantum field theory associated with the stochastic model. Computations have been done in the ladder approximation, which proved to be quite reliable in cases where a rigorous analysis could be performed [6].

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