Existence of Mesons and Mass Splitting in Strong Coupling Lattice QCD

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Keywords: Lattice QCD, Spectral Analysis, Meson Mass Spectrum
Mathematics Subject Classification: 81T25, 81Q10, 46N50

October 20, 2003

Abstract

We consider one flavor lattice QCD in the imaginary time functional integral formulation for space dimensions $d = 2, 3$ with 4×4 Dirac spin matrices, small hopping parameter κ, $0 < \kappa << 1$, and zero plaquette coupling. We determine the energy-momentum spectrum associated with four-component gauge invariant local meson fields which are composites of a quark and an anti-quark field. For the associated correlation functions, we establish a Feynman-Kac formula and a spectral representation. Using this representation, we show that the mass spectrum consists of two distinct masses m_a and m_b, given by $m_c = -2\ln\kappa + r_c(\kappa)$, $c = a, b$, where r_c is real analytic. For $d = 2$, m_a and m_b have multiplicity two and the mass splitting is $\kappa^4 + \mathcal{O}(\kappa^6)$; for $d = 3$, one mass has multiplicity one and the other three, with mass splitting $2\kappa^4 + \mathcal{O}(\kappa^6)$. In the subspace of the Hilbert space generated by an even number of fermion fields the dispersion curves are isolated (upper gap property) up to near the two-meson threshold of asymptotic mass $-4\ln\kappa$.

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1 Introduction

It is fundamental to determine the existence of particles and their spectral properties of quantum field theory. In quantum chromodynamics (QCD), one needs to establish on a rigorous basis the energy-momentum (e-m) spectrum of particles and their bound states, in particular, to prove the existence of mesons and baryons and their bound states. These types of spectral properties involve low-energies and one way to study them is by using a lattice regularization of the continuum, and to consider the strong coupling regime. Strong coupling lattice QCD models give a good insight into the low-energy behavior of QCD. Indeed, our partial understanding of confinement up to now comes in this way, and in fact was one reason for the introduction of lattice QCD (see [1, 2], [3] for a reference book and [4, 5] for recent reviews).

Basically, there have been two routes used in the rigorous studies of the particle spectrum in lattice theories. One is based on methods which are reminiscent of continuum field theories (e.g. decoupling of hyperplanes, Euclidean subtraction) [6, 7, 8, 9, 10, 11] and the other are based on statistical mechanical methods (e.g. random surfaces) [12, 13]. Here, we take the former approach.

As our main result, we prove the existence of meson particles by showing that mesons arise as tightly bound, bound states of a gauge invariant state composed of one quark and one anti-quark. The occurrence of mesons is manifested by an isolated dispersion curve in the e-m spectrum. The same kind of spectral problems that we consider here were also treated in previous works, for other types of particles. For the pure gauge case, with small coupling g_0^{-2}, the low-lying glueball spectrum is found in [8]. The corresponding glueball mass is $\approx 8 \ln g_0$. The determination of the masses in the baryonic sector was considered in [14, 15]. Recently, a rigorous treatment allowed to prove the existence of baryons and their multiplicities for a one-flavor lattice QCD model in the infinitely strong coupling regime and for small enough hopping parameter $\kappa > 0$, $0 < g_0^{-2} \ll \kappa \ll 1$. In [16], this was done for space dimension $d = 2$, using 2×2 Pauli spin matrices, and in [17] for $d = 2, 3$ employing 4×4 Dirac spin matrices. The two-baryon bound state spectrum within the context of [16] was analyzed more recently in [19].

Here, we work with the same lattice QCD model as in [17], but our analysis is restricted to the meson sector of the underlying physical Hilbert space $\mathcal{H}$. The method used in [17] for baryons is largely applied here, with some adaptations. No approximations are made so our results are exact within this context. We will assume the results of [17] regarding positivity and the construction of $\mathcal{H}$. We begin directly by introducing a local four-component meson field. For the associated correlation functions (cf's), we establish a Feynman-Kac formula and a spectral representation analogous to the Källen-Lehman representation in quantum field theory. We emphasize that it is this representation that allows us to identify complex momentum space singularities of the two-point function with points in the e-m spectrum. To our knowledge, such a representation as given here is not found in the literature. We also point out that determining exponential decay rates of cf's, as in [13], is *not sufficient* to identify these rates with the mass spectrum, since a connection with the energy-momentum spectrum is not established. This is especially true in the case where there are multiplicities involved and mass splitting, as here.

Let $\mathcal{H}_e \subset \mathcal{H}$ denote the subspace generated by an even number of fermions. We show that there are meson particles, with asymptotic masses $-2\ln\kappa$, manifested by isolated dispersions curves in the e-m spectrum (upper mass gap property) and it is the *only* spectrum in the subspace $\mathcal{H}_e \subset \mathcal{H}$, up to near the two-meson threshold which is asymptotically of order $-4\ln\kappa$.

Besides its intrinsic importance, the determination and control of the meson spectrum is an essential step towards the understanding of e.g. meson-meson and meson-baryon bound states from first principles. Indeed, a two-meson bound state analysis is in progress [20], using e.g. our techniques from previous works (see [10, 11], and [19]). Also, our results open the way to study the e-m spectrum for the more realistic case when the glueball mass is large, but is such that $g_0^{-2} \ll 1$ cannot be neglected in comparison with the small hopping parameter $\kappa \ll 1$.

2 The Results

We first give a very brief definition of one flavor SU(3) gauge-matter QCD model of [17] and present our results. We use the same notation as in [17], and the reader is referred there for all details concerning notation, the definition of the model, the introduction of energy and momentum operators, the derivation of a Feynman-Kac type formula for general correlation functions, the construction of the underlying physical Hilbert space $\mathcal{H}$, the computation of gauge and Fermi (Grassmann) integrals, and the analysis of symmetries other than gauge that is the main result of its Theorem 4, namely, the symmetries of charge conjugation, parity, lattice coordinate reflections, and lattice spatial rotations. We omit these and other details here, for the sake of brevity.

We consider the case where the space dimension is $d = 2, 3$ and the gauge group is SU(3). The partition function of the model is given formally by

$$Z = \int e^{-S(\psi,\bar\psi,g)}\, d\psi\, d\bar\psi\, d\mu(g)\ , \tag{2.1}$$

and for a function $F(\bar\psi, \psi, g)$, of the single flavor (anti-)quark Grassmann fields ψ and $\bar\psi$, and the gauge field g, the normalized expectations, in the thermodynamic limit, are denoted by $\langle F(\bar\psi, \psi, g)\rangle \equiv \frac{1}{Z}\int F(\bar\psi, \psi, g) e^{-S(\psi,\bar\psi,g)} d\psi d\bar\psi d\mu(g)$. The gauge invariant action $S(\psi, \bar\psi, g)$ is given by

$$\begin{aligned} S(\psi,\bar\psi,g) = \quad & \frac{\kappa}{2}\sum \bar\psi_{a\alpha}(u)\, \Gamma^{\epsilon e^\rho}_{\alpha\beta}\, (g_{u,u+\epsilon e^\rho})_{ab}\, \psi_{b\beta}(u+\epsilon e^\rho) \\ & + \sum_{u\in\mathbb{Z}_o^{d+1}} \bar\psi_{a\alpha}(u) M_{\alpha\beta}\psi_{a\beta}(u) - \frac{1}{g_0^2}\sum_p \chi(g_p), \end{aligned} \tag{2.2}$$

where the first sum runs over $u = (u^0, \vec{u}) = (u^0, u^1, \ldots, u^d) \in \mathbb{Z}_o^{d+1} \equiv \mathbb{Z}_{1/2} \times \mathbb{Z}^d$, where $\mathbb{Z}_{1/2} = \{\pm 1/2, \pm 3/2, \ldots\}$, $\epsilon = \pm 1$, $\rho = 0, 1, \ldots, d$, and over repeated indices. For notational simplicity, we sometimes drop U from the gauge matrix $U(g)$ associated with each oriented lattice bond. Concerning the parameters, we take $m > 0$, $0 < g_0^{-2} \ll \kappa \ll 1$ and $M \equiv M(m,\kappa) = (m+2\kappa)I_4$, I_4 being the 4×4 identity matrix. Also, within the family of actions of [2], we have $\Gamma^{\pm e^\rho} = -1 \pm \gamma^\rho$. For $d=3$, γ^ρ are the 4×4 hermitian traceless anti-commuting Dirac matrices. For $d=2$, all but the γ^3 matrix appear in the action. The measure $d\mu(g)$ is the product measure over non-oriented bonds of normalized SU(3) Haar measures (see [22]) and the integrals over Grassmann fields are defined according to [21]. In Eq. (2.1), $d\psi\, d\bar\psi$ means $\prod_{u,a,\alpha} d\psi_{a\alpha}(u)\, d\bar\psi_{a\alpha}(u)$ such that, with a normalization $\mathcal{N} = \langle 1\rangle$, we have $\langle \psi_{a\alpha}(x)\,\bar\psi_{b\beta}(y)\rangle = (1/\mathcal{N}) \int \psi_{a\alpha}(x)\,\bar\psi_{b\beta}(y)\, e^{-\sum_u \bar\psi_{a'\alpha'}(u) M_{\alpha'\beta'}\psi_{a'\beta'}(u)}\, d\psi\, d\bar\psi = M^{-1}_{\alpha\beta}\delta_{ab}\delta(x,y)$, with a Kronecker delta for space-time coordinates. With our restrictions on the parameters, there is a quantum mechanical Hilbert space $\mathcal{H}$ of physical states, for $\kappa > 0$; and the condition $m > 0$ guarantees that the one-particle free Fermion dispersion curve increases in each positive momentum component. Last, without loss of generality, we set $M = I_4$ in (2.2).

We note that by polymer expansion methods (see [2, 9, 23]), correlation functions (cf's) exist and are lattice translational invariant in the thermodynamic limit and truncated correlations have exponential tree decay. Furthermore, the cf's extend to analytic functions in the coupling parameters κ and g_0^{-2}.

To state our results on the existence of meson particles, their masses and dispersion curves, we consider the subspace $\mathcal{H}_m \subset \mathcal{H}$ generated by vectors associated with the gauge invariant meson fields composed of a fermion (quark) and an anti-fermion (anti-quark) given by

$$\mu_\alpha(u) = \begin{cases} \frac{1}{\sqrt6}(\psi_{a3}(u)\bar\psi_{a1}(u) + \psi_{a4}(u)\bar\psi_{a2}(u)) & , \quad \alpha = 1 \\ \frac{1}{\sqrt3}\psi_{a4}(u)\bar\psi_{a1}(u) & , \quad \alpha = 2 \\ \frac{1}{\sqrt3}\psi_{a3}(u)\bar\psi_{a2}(u) & , \quad \alpha = 3 \\ \frac{1}{\sqrt6}(\psi_{a3}(u)\bar\psi_{a1}(u) - \psi_{a4}(u)\bar\psi_{a2}(u)) & , \quad \alpha = 4\,; \end{cases} \tag{2.3}$$

$$\pi_\beta(v) = \begin{cases} \frac{1}{\sqrt6}(\bar\psi_{a3}(v)\psi_{a1}(v) + \bar\psi_{a4}(v)\psi_{a2}(v)) & , \quad \beta = 1 \\ \frac{1}{\sqrt3}\bar\psi_{a4}(v)\psi_{a1}(v) & , \quad \beta = 2 \\ \frac{1}{\sqrt3}\bar\psi_{a3}(v)\psi_{a2}(v) & , \quad \beta = 3 \\ \frac{1}{\sqrt6}(\bar\psi_{a3}(v)\psi_{a1}(v) - \bar\psi_{a4}(v)\psi_{a2}(v)) & , \quad \beta = 4\,. \end{cases} \tag{2.4}$$

The normalization is chosen so that the associated matrix-valued two-point cf (see below), at coincident points and zero hopping parameter κ, equals I_4.

To see the connection with the e-m spectrum, we define a two-point cf, establish a Feynman-Kac formula and a spectral representation. The normalized two-point cf is defined by (χ here denotes the characteristic function)

$$\begin{aligned} G_{\alpha\beta}(u,v) &= \langle \mu_\alpha(u)\pi_\beta(v)\rangle_T\, \chi_{u^0\le v^0} + \langle \pi_\alpha(u)\mu_\beta(v)\rangle_T^*\, \chi_{u^0> v^0} \\ &= \langle \mu_\alpha(u)\pi_\beta(v)\rangle\, \chi_{u^0\le v^0} + \langle \pi_\alpha(u)\mu_\beta(v)\rangle^*\, \chi_{u^0> v^0} = G_{\alpha\beta}(u-v), \end{aligned}$$

where the truncation $\langle\ \rangle_T$ is given by

$$\langle F(u)H(v)\rangle_T = \langle F(u)H(v)\rangle - \langle F(u)\rangle\,\langle H(v)\rangle\,, \tag{2.5}$$

and we used that the truncation for $G_{\alpha\beta}$ is zero, by parity symmetry.

This seemingly awkward definition has three desirable features. First, the extension to equal times of the $u^0 > v^0$ definition agrees with the one for $u^0 \le v^0$, by translational invariance and time reversal. Second, its Fourier transform admits a simple spectral representation. Third, it permits us to show the existence of particles, such as the upper mass gap property.

We set $\mathcal{E}(\lambda^0,\vec{\lambda}) = \mathcal{E}_0(\lambda^0)\mathcal{E}(\vec{\lambda})$, where $\mathcal{E}_0(\lambda^0)$ is the spectral family for the time translation operator T_0, and $\mathcal{E}(\vec{\lambda}) = \prod_{i=1}^d \mathcal{E}_i(\lambda^i)$ is the product of the spectral families for the i-th component self-adjoint momentum operator P^i which generates lattice space translations along the i-th direction e^i, $i = 1,\dots d$. The spectral representation of the next proposition is an important tool. Its proof is omitted here since it follows closely the one for baryons given in [17].

Proposition 2.1 *For $u^0 \neq v^0$, $\pi_\alpha \equiv \pi_\alpha(1/2,\vec{0})$, the following F-K formula holds for the two-point meson cf (and the r.h.s. is an even function of $\vec{v}-\vec{u}$):*

$$G_{\alpha\beta}(u,v) \quad = \quad \int_{-1}^{1}\int_{\mathbf{T}^d} (\lambda^0)^{|v^0-u^0|-1} e^{i\vec{\lambda}.(\vec{v}-\vec{u})} d(\pi_\alpha, \mathcal{E}(\lambda^0,\vec{\lambda})\pi_\beta)_{\mathcal{H}}\,. \tag{2.6}$$

We now obtain a spectral representation for the Fourier transform of the two point function $G_{\alpha\beta}(u,v)$. For $x\in\mathbb{Z}^{d+1}$, with an abuse of notation, we define $G_{\alpha\beta}(x = u-v) \equiv G_{\alpha\beta}(u,v)$. Then, the Fourier transform $\tilde{G}_{\alpha\beta}(p) = \sum_{x\in\mathbb{Z}^{d+1}} G_{\alpha\beta}(x)e^{-ipx}$, $p\in\mathbf{T}^{d+1}$, admits the spectral representation

$$\begin{aligned}\tilde{G}_{\alpha\beta}(p) \;=\; \tilde{G}_{\alpha\beta}(\vec{p}) + (2\pi)^d \int_{-1}^{1}\int_{\mathbf{T}^d} \delta(\vec{p}-\vec{\lambda})\frac{2(\cos p^0-\lambda^0)}{1+(\lambda^0)^2-2\lambda^0\cos p^0} \\ \times\, d_{\lambda^0} d_{\vec{\lambda}}(\pi_\alpha(1/2,\vec{0}),\mathcal{E}_0(\lambda^0)\mathcal{E}(\vec{\lambda})\pi_\beta(1/2,\vec{0}))_{\mathcal{H}}\,,\end{aligned} \tag{2.7}$$

where $\tilde{G}(\vec{p}) = \sum_{\vec{x}\in\mathbb{Z}^d} e^{-i\vec{p}.\vec{x}}G(x^0=0,\vec{x})$. From the above spectral representation, we see that points of non-analyticity in p^0, on the imaginary axis, are points in the e-m spectrum. It is possible that points of non-analyticity of the form $p^0 = \pm\pi + iq^0$ can occur but this is shown not to be the case in our analysis. We determine the spectrum and show the existence of isolated dispersion curves, up to near the threshold $-4\ln\kappa$, by showing that $\Gamma_{\alpha\beta}(u,v)$, the convolution inverse of the two-point function $G_{\alpha\beta}(u,v)$, decays faster than $G_{\alpha\beta}(u,v)$, and hence the Fourier transform $\tilde{\Gamma}_{\alpha\beta}(p)$ of $\Gamma_{\alpha\beta}(x=u-v) = \Gamma_{\alpha\beta}(u,v)$ has a larger region of analyticity in p^0. Thus, as $\tilde{G}(p)\tilde{\Gamma}(p) = I_4$, $\tilde{\Gamma}^{-1}_{\alpha\beta}(p) = [\mathrm{cof}\,\tilde{\Gamma}]_{\beta\alpha}(p)/\det[\tilde{\Gamma}(p)]$ provides a meromorphic extension of $\tilde{G}_{\alpha\beta}(p)$, and the e-m spectrum occurs, for each $\vec{p}$, as points given by the p^0 imaginary axis zeroes of $\det[\tilde{\Gamma}(p)]$.

The reader may wonder, e.g. for space dimension $d=3$, why composite fields of the form $\bar{\psi}(u)\gamma^5\psi(u)$ and, for $k=1,2,3$, $\bar{\psi}(u)\gamma^k\psi(u)$, $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, are not employed to form the two-point cf matrix rather than those of Eqs. (2.3) and

(2.4). Intuitively, these are the vector and the pseudoscalar meson fields. The reason is that, with our choices, it turns out that the zero spatial momentum matrix $\tilde{G}_{\alpha\beta}(p^0, \vec{p} = \vec{0})$ is already diagonal (see Section 4, Lemma 4.1), so that the masses m_α are determined as the solutions of $\tilde{\Gamma}_{\alpha\alpha}(p^0 = im_\alpha, \vec{p} = \vec{0}) = 0$, $\alpha = 1, 2, 3, 4$.

The zero space momentum improper states we obtain can be classified using the irreducible discrete Z_4 rotation group generated by $\pi/2$ rotations in the x^1x^2-plane and the group Z_2 generated by x^3-coordinate reflections. Using the results of Theorem 4 of ref. [17], about symmetries, we find that, for $\pi_i' \equiv \sum_{\vec{x}\in\mathbb{Z}^{d-1}} \pi_i(1/2, \vec{x})$, the improper zero momentum states π_i', $i = 1, 2, 3, 4$, satisfy: π_1' and π_4' form a basis for the trivial representation of Z_4, while π_2' (π_3') is a basis for the representation of Z_4 generated by i $(-i)$ times the identity; $\pi_1' + \pi_4'$ ($\pi_1' - \pi_4'$) forms a basis for the trivial (non-trivial) representation of Z_2; and π_2' and π_3' form a basis for the trivial representation of Z_2. The commuting groups Z_4 and Z_2 are of course subgroups of the full lattice rotation and reflection group. We note that all the states have parity (-1), so that parity does not distinguish among them.

To analyze $\det \tilde{\Gamma}(p)$, it suffices to obtain a long range bound for $\Gamma(x)$, but we need its precise short distance behavior, for $|x| \leq 2$ to determine the masses and the mass splitting up to and including the order κ^4. However, to control the error, bounds on $\Gamma(x)$ (which improve those obtained by the hyperplane decoupling method) are needed for some x's, with $|x^0| \leq 3$. These bounds are obtained by exhibiting cancellations in the Neumann series.

The two-point function convolution inverse $\Gamma(x)$ is defined by the Neumann series $\Gamma = \sum_{i=0}^{\infty} (-1)^i [G_d^{-1} G_n]^i G_d^{-1}$, where G_d is the diagonal part of G

$$G_{d,\alpha\beta}(u, v) \quad = \quad G_{\alpha\alpha}(u, u)\delta_{\alpha\beta}\delta(u, v) \tag{2.8}$$

and G_n is the remainder $G_{n,\alpha\beta}(u, v) = G_{\alpha\beta}(u, v) - G_{d,\alpha\beta}(u, v)$. By the bounds in Theorems 2.1 and 2.2 below, G, G_d, G_n and Γ are bounded as matrix operators on $\ell_2(\mathbb{C}^4 \times \mathbb{Z}_o^{d+1})$.

Moreover, $\Gamma_{\alpha\beta}(x)$ is analytic in κ as $G_{\alpha\beta}(x)$ is, and its short distance behavior is determined by expanding in κ. Long-range bounds on the decay of $G_{\alpha\beta}(x)$ and $\Gamma_{\alpha\beta}(x)$ are obtained by the decoupling of the hyperplane method (see [6, 7, 9, 11]). Our results hold for all sufficiently small hopping parameter $\kappa > 0$, $0 < g_0^{-2} \ll \kappa \ll 1$. The short-distance behavior and bounds on G and Γ are given in the next two theorems.

Theorem 2.1 *Let $0 < g_0^{-2} \ll \kappa \ll 1$, c be a positive constant, $\rho, \sigma = 0, 1, \ldots, d$ and e^0 denote the time unit vector, e^i, e^j $(i, j = 1, \ldots, d)$, denote space unit vectors, $|\vec{x}| \equiv \sum_{i=1}^{d} |x^i|$ and $\epsilon, \epsilon' = \pm 1$. The following properties hold for G:*

1.

$$G_{\alpha\beta}(x) \;=\; \begin{cases} \delta_{\alpha\beta} + \mathcal{O}(\kappa^8) & , \quad x = 0\,; \\ \delta_{\alpha\beta}\kappa^2 + \mathcal{O}(\kappa^6) & , \quad x = \epsilon e^0\,; \\ c_2\delta_{\alpha\beta}\kappa^2 + \mathcal{O}(\kappa^6) & , \quad x = \epsilon e^j\,; \\ \delta_{\alpha\beta}\kappa^4 + \mathcal{O}(\kappa^8) & , \quad x = 2\epsilon e^0\,; \\ c_2\delta_{\alpha\beta}\kappa^4 + \mathcal{O}(\kappa^8) & , \quad x = 2\epsilon e^j\,; \\ c_{\alpha\beta}(x)\kappa^4 + \mathcal{O}(\kappa^8) & , \quad x = \epsilon e^\rho + \epsilon' e^\sigma\ ,\ \rho < \sigma\,. \end{cases} \tag{2.9}$$

where the κ independent constants are given by $c_2 = 1/4$, and

$$c_{\alpha\beta}(x) \;=\; \begin{cases} \delta_{\alpha\beta}/2\ ,\ x = \epsilon e^0 + \epsilon' e^\rho \\ c^{ij}_{\alpha\beta}\ ,\ x = \epsilon e^i + \epsilon' e^j,\ i < j \end{cases}\ , \tag{2.10}$$

where $c^{ij}_{\alpha\beta}$ is given by: $c^{12}_{\alpha\beta}$ is diagonal with $4c^{12}_{11} = 4c^{12}_{44} = 1$ and $c^{12}_{22} = c^{12}_{33} = 0$, $4c^{13}_{11} = 8c^{13}_{22} = 8c^{13}_{33} = -8c^{13}_{23} = -8c^{13}_{32} = 1$, $4c^{23}_{11} = 8c^{23}_{22} = 8c^{23}_{33} = 8c^{23}_{23} = 8c^{23}_{32} = 1$, and the remaining elements are zero.

2. $$|G_{\alpha\beta}(x)| \;\leq\; c|\kappa|^{2|x^0|+2|\vec{x}|}\,. \tag{2.11}$$

3. $G_{\alpha\beta}(x) = d_\alpha G_{\alpha\beta}(x') d^*_\beta$, *where*

   (a) *for $x' = (x^0, -x^2, x^1, x^3)$, $d_1 = 1$, $d_2 = -i$, $d_3 = i$, $d_4 = 1$;*

   (b) *for $x' = (x^0, -x^1, -x^2, x^3)$, $d_1 = 1$, $d_2 = -1$, $d_3 = -1$, $d_4 = 1$;*

   (c) *for $x' = (x^0, -x^1, -x^2, -x^3)$, $d_\alpha = -1$, for $\alpha = 1, 2, 3, 4$.*

   Also,
   $$G(e^1 + e^2) = SG(e^1 + e^3)S^\dagger,$$
   where $S = S^t$ and $S^{-1} = S^\dagger$ (superscript t means transpose and $\dagger$ Hermitian conjugate). S has the matrix elements $S_{11} = 1$, $S_{12} = S_{13} = S_{14} = S_{44} = 0$, $S_{22} = S_{33} = S_{23} = 1/2$ and $S_{24} = S^*_{34} = i\sqrt{2}/2$.

Remark 2.1 *The equality of the diagonal elements of $c^{13}_{\alpha\beta}$ and $c^{23}_{\alpha\beta}$ follows from the symmetry results of Theorem 4 in [17] below, namely rotations by $\pi/2$ about the e^3 axis.*

Remark 2.2 *The third result above follows by using the symmetry results of Theorem 4 in [17] below and holds to all orders in κ.*

The short-distance and decay behaviors of $\Gamma(x)$ are given by:

Theorem 2.2 *Under the same hypotheses of Theorem 2.1, with c_2 and $c_{\alpha\beta}(x)$ as given there, we have:*

1.

$$\Gamma_{\alpha\beta}(x) = \begin{cases} \delta_{\alpha\beta} + (2+2dc_2^2)\delta_{\alpha\beta}\kappa^4 + \mathcal{O}(\kappa^8)\ ,\ x=0\,; \\ -\delta_{\alpha\beta}\kappa^2 + \mathcal{O}(\kappa^8)\ ,\ x=\epsilon e^0\,; \\ -c_2\delta_{\alpha\beta}\kappa^2 + \mathcal{O}(\kappa^6)\ ,\ x=\epsilon e^j\,; \\ \mathcal{O}(\kappa^{10})\ ,\ x=2\epsilon e^0\,; \\ (-c_2+c_2^2)\,\delta_{\alpha\beta}\kappa^4 + \mathcal{O}(\kappa^8)\ ,\ x=2\epsilon e^j\,; \\ \left[-c_{\alpha\beta}(x) + 2c_2^2\delta_{\alpha\beta}\right](1-\delta_{0\rho})(1-\delta_{0\sigma})\kappa^4 + \mathcal{O}(\kappa^8)\ , \\ \qquad\qquad\qquad x=\epsilon e^\rho + \epsilon' e^\sigma\ ,\rho<\sigma\,; \\ \mathcal{O}(\kappa^8)\ ,\ |x^0|=1,\ |\vec{x}|=2\,; \\ \mathcal{O}(\kappa^{10})\ ,\ |x^0|=2,\ |\vec{x}|=1\,; \\ \mathcal{O}(\kappa^{12})\ ,\ |x^0|=3,\ |\vec{x}|=0\,. \end{cases} \tag{2.12}$$

2.
$$|\Gamma_{\alpha\beta}(x)| \leq \begin{cases} c|\kappa|^{2+4(|x^0|-1)+2|\vec{x}|}\ ,\ |x^0|>1\,; \\ c|\kappa|^{2|x^0|+2|\vec{x}|}\ ,\ |x^0|\leq 1\,. \end{cases} \tag{2.13}$$

Remark 2.3 *The absence of lower order terms in $\Gamma_{\alpha\beta}$, as compared to $G_{\alpha\beta}$, for $|x^0| = 1,2,3$, is due to explicit cancellations in the Neumann series and improves the hyperplane method bounds obtained in Theorem 2.2 , item two.*

Concerning the mass spectrum, i.e. the e-m spectrum at zero-space momentum, it turns out that the mass is determined up to $\mathcal{O}(\kappa^4)$ by the values of $\Gamma_{\alpha\beta}(x)$ up to distance $|x|\leq 2$, and $\alpha=\beta$. The κ^4 contribution to $\Gamma(x)$, for these values of x, comes from the first and second order terms in G_n in the Neumann series. The second order terms are independent of α since they are products of two κ^2 terms of $G_{n,\alpha\beta}(x)$, for points x of distance one, which are diagonal and independent of α. For the first order term in $G_n(x)$, $|x|=2$, the κ^4 contributions come from *straight* contributions and *angle* contributions. Straight contributions have two subsequent sets of two bonds, with opposite orientation, connecting e.g. the point 0 to ϵe^ρ and then to $2\epsilon e^\rho$; and the property of the Γ matrices given above guarantees that these contributions behave like the κ^2, diagonal and α-independent contributions $G_{n,\alpha\beta}(x)$, $|x|=1$, and do not give rise to mass splitting as well. Angles are contributions to $G_{n,\alpha\beta}$ for points of the form $x=\epsilon e^i+\epsilon' e^j$, $i,j=1,2,\ldots,d$, $i<j$, $\epsilon,\epsilon'=\pm1$. These are L-type contributions associated with two sets of two lattice bonds, with opposite orientation; one set connecting the points $0\to\epsilon e^i$ and the other connecting $\epsilon e^i\to\epsilon e^i+\epsilon' e^j$, or one set connecting the points $0\to\epsilon' e^j$ and the other connecting $\epsilon' e^j\to\epsilon e^i+\epsilon' e^j$. They contribute to mass splitting in $\mathcal{O}(\kappa^4)$ for $d=2$, 3. (See Theorem 2.3 .)

Before stating our results on the mass spectrum and dispersion curves, we give an intuitive picture for the *asymptotic* behavior of the mass. Retaining only the diagonal part of $\Gamma_{\alpha\beta}(x)$, $x=0$ and $x=(1,\vec{0})$, the equation for the mass m is

$$\det\tilde{\Gamma}(p^0=im,\vec{p}=\vec{0}) \approxeq (1-\kappa^2\,e^m)^4 = 0\,,\ m\in\mathbb{R}\,,$$

so we have a mass m of order $-2\ln\kappa$, with a four-fold degeneracy.

To rigorously determine the e-m spectrum, we exploit symmetries for $\vec{p}=\vec{0}$. $\tilde{\Gamma}(p^0 = im, \vec{p}=\vec{0})$ is shown to be diagonal. The determination of the non-singular part can be cast into an analytic implicit function problem. For $\vec{p}\neq\vec{0}$, we have not shown that $\tilde{\Gamma}(p^0 = im, \vec{p})$ is diagonal, but the asymptotic form of the dispersion curve can be obtained by a Rouché's theorem argument for the zeroes of $\det\tilde{\Gamma}(p^0 = iw(\vec{p}), \vec{p})$. Also, our results in $\mathcal{H}_m$ are extended to the whole Hilbert space $\mathcal{H}_e$ by adapting the subtraction method of [8].

The results for the e-m spectrum are given in the theorem below.

Theorem 2.3 *Under the hypotheses of Theorem 2.1 , the following mesonic spectral results hold in the even subspace $\mathcal{H}_e$ of the physical Hilbert space $\mathcal{H}$.*

1. *To any order in κ and for $d=2,3$, the mass spectrum in $\mathcal{H}_e$ and in the energy interval $(0,-(4-\epsilon)\ln\kappa)$, $\epsilon>0$, consists of three masses given by $diag[a+c,b,b,a-c]$, $a+c$ for $\alpha=1$, b for $\alpha=2,3$ and $a-c$ for $\alpha=4$. Up to and including order κ^4, $a-c=b$, for $d=3$, and $c=0$, for $d=2$, and there are only two distinct masses m_a and m_b given by*

$$m_j = -2\ln\kappa + r_j(\kappa),\ \ j=a \ \text{or}\ \ j=b,$$

   where $r_j(\kappa)\equiv\sum_{n=2}^{\infty} b_{c,n}\kappa^n$ is real analytic in κ at 0, for each $d=2,\,3$. We obtain, with the constants given in Theorem 2.1 ,

$$\begin{aligned} r_j(\kappa) &= -2dc_2\kappa^2 + [-4(c^{12}_{\alpha\alpha}+c^{13;23}_{\alpha\alpha}) \\ &\quad +(1-2dc_2-24c_2^2+20dc_2^2)-2d^2c_2^2]\kappa^4 + \mathcal{O}(\kappa^6)\,, \end{aligned}$$

   where $c^{13;23}_{\alpha\alpha}\equiv c^{13}_{\alpha\alpha}+c^{23}_{\alpha\alpha}$ is to be omitted for $d=2$. Again, up to and including order κ^4, for $d=3$, the mass m_b is associated with $\alpha=1$ and has multiplicity one; m_a is associated with $\alpha=2,3,4$, and has multiplicity three. The mass splitting is $m_a-m_b=2\kappa^4+\mathcal{O}(\kappa^6)$. For $d=2$, m_b (m_a) is associated with $\alpha=1,4$ ($\alpha=2,3$), and both have multiplicity two. The mass splitting is $m_a-m_b=\kappa^4+\mathcal{O}(\kappa^6)$.

2. *The e-m spectrum in $\mathcal{H}_e$ and in the energy interval $(0,-(4-\epsilon)\ln\kappa)$, $\epsilon>0$, consists of four dispersion curves (not necessarily distinct), each of which has the form*

$$w(\vec{p}) = -2\ln\kappa - 2dc_2\kappa^2 + c_2\kappa^2\sum_{j=1}^{d} 2(1-\cos p^j) + \mathcal{O}(\kappa^4). \qquad (2.14)$$

   The curves $w(\vec{p})$ are increasing functions of each component p^j of $\vec{p}$, and are convex for small $|\,\vec{p}\,|$.

Remark 2.4 *The action of the charge conjugation transformation defined below leaves the space $\mathcal{H}_m$ stable, and hence we have the same spectral representation for particles and anti-particles, as given by Eq. (2.7). At the level of cf's, the two-point function for the meson anti-particle (which we call G'), is related to*

G by $G' = TGT^{-1}$, with $T = diag[1,-1,-1,-1]$, for $d = 3$. For $d = 2$, replace T by U, where $U = diag[-1,-1,-1,1]$. As the mass and dispersion curves are determined by the implicit equation $\det \tilde{\Gamma}(p_0, \vec{p}) = 0$*, the meson particle and anti-particle mass spectrum and dispersion curves are identical.*

The organization of the rest of the paper is also patterned by that in [17]. Section 3 is devoted to the proof of Theorems 2.1 and 2.2 . In Section 4, we prove Theorem 2.3 .

3 Decay Bounds and Short-Distance Behavior of G and Γ

We now use the decoupling of hyperplane method to obtain bounds on G and Γ, as given in Theorem 2.1 . We assume that the reader has some familiarity with this method and refer to [6, 9, 11] for more details.

We will encounter gauge group integrals of monomials in the group matrix elements g_{ij} ($i,j = 1,2,3$) denote SU(3) matrix elements, and we suppress the lattice points from the notation) and the inverses g^{-1}_{ij}. These are evaluated following techniques developed in the Chapter 8 of [24] and in [25, 26], for the general SU(N) case. In particular, to prove Theorem 2.1 , we will need $\int g_{a_1b_1} g^{-1}_{a_2b_2}\, d\mu(g) = \frac{1}{3}\delta_{a_1b_2}\delta_{a_2b_1}$ and $\int g_{a_1b_1} g^{-1}_{a_2b_2} g_{a_3b_3} g^{-1}_{a_4b_4}\, d\mu(g) = \frac{1}{8}(\delta_{a_1b_2}\delta_{a_3b_4}\delta_{b_1a_2}\delta_{b_3a_4} + \delta_{a_1b_4}\delta_{a_3b_2}\delta_{b_1a_4}\delta_{b_3a_2}) - \frac{1}{24}(\delta_{a_1b_2}\delta_{a_3b_4}\delta_{b_1a_4}\delta_{b_3a_2} + \delta_{a_1b_4}\delta_{a_3b_2}\delta_{b_1a_2}\delta_{b_3a_4})$. Also, we use the following properties involving Γ matrices ($\rho, \sigma = 0,1,\ldots,d$, and $\epsilon,\ \epsilon' = \pm 1$) $\Gamma^{\epsilon e^\rho}\Gamma^{-\epsilon e^\rho} = 0$, $\Gamma^{\epsilon e^\rho}\Gamma^{\epsilon e^\rho} = -2\Gamma^{\epsilon e^\rho}$ and $\Gamma^{\epsilon e^\rho}\Gamma^{\epsilon' e^\sigma} = 2I_4 - \Gamma^{-\epsilon' e^\sigma}\Gamma^{-\epsilon e^\rho}$.

We will obtain decay properties for the gauge invariant truncated cf's

$$\langle F(u)H(v)\rangle_T = \langle [T_0^{u^0-1/2}\vec{T}^{\vec{u}} F(1/2,\vec{0})][T_0^{v^0-1/2}\vec{T}^{\vec{v}} H(1/2,\vec{0})]\rangle_T \tag{3.1}$$

where $F, H \in \mathcal{H}_e$, T_0 is time translation by e^0 and $\vec{T}^{\vec{u}} = T_1^{u^1}\ldots T_d^{u^d}$ is space translation by $\vec{u} = (u^1,\ldots,u^d) \in \mathbb{Z}^d$.

We discuss explicitly the decoupling procedure for the time (vertical) direction. The space directions are treated similarly, and together with the time direction. The arguments are carried out for cf's in a finite volume $\Lambda \in \mathbb{Z}_o^{d+1}$, with bounds uniform in the volume $|\Lambda|$, and extend to the infinite volume using standard consequences of the polymer expansion (see e.g. [9, 23]).

For $u^0 < v^0$, $p \in \mathbb{Z}$, $u^0 + 1/2 \le p \le v^0 - 1/2$ (or, if $u^0 > v^0$, $v^0 + 1/2 \le p \le u^0 - 1/2$), replace the hopping parameter $\kappa > 0$ multiplying the nonlocal fermionic part of the action (2.2) (*not* the κ in M) by $\kappa_p \in \mathbb{C}$ and denoting $\partial^r/\partial\kappa_p^r$ by ∂^r and by ∂_0^r its $\kappa_p = 0$ value, the following properties hold.

Lemma 3.1 *Concerning the derivatives of G, we have:*

1. *If $u^0 \neq v^0$, $\partial_0^r\langle F(u)H(v)\rangle_T = 0$, $r = 0,1,3$.*

2. *If $u^0 < v^0$, and $r = 2$,*

$$\partial_0^2 \langle F(u) H(v) \rangle_T \,=\, 2 \sum_{w | w^0 = -1/2+p} \langle F(u) \pi_\alpha(w) \rangle_T \langle \mu_\alpha(w+e^0) H(v) \rangle_T \, \big|_{\kappa_p = 0} .$$

3. *If $u^0 > v^0$, and $r = 2$,*

$$\partial_0^2 \langle F(u) H(v) \rangle_T \,=\, 2 \sum_{w | w^0 = -1/2+p} \langle F(u) \mu_\alpha(w+e^0) \rangle_T \langle \pi_\alpha(w) H(v) \rangle_T \, \big|_{\kappa_p = 0} .$$

Proof . Consider $u^0 < v^0$; the argument for $v^0 < u^0$ is similar. For the truncated two-point function of Eq. (2.5), we introduce a duplicate variable representation (see [8]) depending on the hyperplane decoupling parameters $\{\kappa_p\}$

$$\begin{aligned} \langle F(u) H(v) \rangle_T = & \frac{1}{2Z^2} \int \left[F(u) - F'(u) \right] \left[H(v) - H'(v) \right] \\ & \times \, \exp \left[- \textstyle\sum_{w | w_0 = -1/2+p} \kappa_p (A(\psi, \bar\psi, g, w) + A(\psi', \bar\psi', g', w)) \right] \\ & \times \, \exp \left[-S(\psi, \bar\psi, g) - S(\psi', \bar\psi', g') \right] \, d\psi \, d\bar\psi \, d\mu(g) \, d\psi' \, d\bar\psi' \, d\mu(g') \, , \end{aligned} \tag{3.2}$$

where $A(\psi, \bar\psi, g, w) = \sum_\epsilon \bar\psi_{a\alpha}(w) \, \Gamma^{\epsilon e^0}_{\alpha\beta} (g_{w, w+\epsilon e^0})_{ab} \, \psi_{b\beta}(w + \epsilon e^0)/2$ and Z^2 is the normalization factor. The primes in F' and H' mean functions of the duplicate variables ψ', $\bar\psi'$ and g'. $S(\psi, \bar\psi, g)$ is the action for the remaining bonds. We now expand the numerator and denominator of (3.2) in powers of κ_p. For the denominator the κ_p coefficient is a sum of bond terms and each term is a product of expectations containing a single ψ or $\bar\psi$ which Fermi integrates to zero. The coefficient of κ_p^2 is a sum of terms with two bounds which must be coincident and of opposite orientation to give a non-zero contribution. For the numerator, we consider the coefficients of κ_p. For the first statement, the κ_p^0 is trivially zero. For κ_p^1 and κ_p^3 the expectation factorizes and each factor has an odd number of fermions and gives zero. The integral over inter-hyperplane gauge field could also be used to show zero contribution for κ_p^1. For the second statement, the κ_p^2 coefficient has terms with two bonds. For a non-zero contribution performing the gauge integral and the matrix structure of $\Gamma^{\pm e^0}$ gives the result. The proof of the third statement is similar to the second one. □

To calculate the κ_p derivatives of Γ, it is convenient to consider instead $\Gamma' \equiv -\Gamma$, minus the convolution inverse of G, and use the formula

$$\partial^r \Gamma' = \sum_{s=0}^{r-1} \begin{pmatrix} r \\ s \end{pmatrix} \Gamma' \, \partial^{r-s} G \, \partial^s \Gamma' \, . \tag{3.3}$$

The first three $\kappa = 0$ derivatives of Γ are given in the next lemma.

Lemma 3.2 *For the derivatives of Γ, we have:*

1. *If $u^0 \neq v^0$, $\partial_0^r \Gamma(u, v) = 0$, $r = 0, 1$.*

2. *If* $u^0 < v^0$, $\partial_0^2\Gamma(u,v) = -2\sum_{w|w^0=-1/2+p}\ \delta(u,w)\delta(w+e^0,v)$.
3. *If* $u^0 > v^0$, $\partial_0^2\Gamma(u,v) = -2\sum_{w|w^0=-1/2+p}\ \delta(u,w+e^0)\delta(w,v)$.
4. *If* $|u^0 - v^0| > 1$, $\partial_0^3\Gamma(u,v) = 0$.

Proof . For the first statement, consider first $u^0 < v^0$. Using Eq. (3.3) and Lemma 3.1 , the result follows directly; and similarly for $u^0 > v^0$. For the second statement, using the first one and the second statement of Lemma 3.1 , we have

$$\partial_0^2\Gamma(u,v) = - \sum_{w,z|w^0+1/2\le p\le z^0-1/2} \Gamma(u,w)\partial^2 G(w,z)\Gamma(z,v)\Big|_{\kappa_p=0} = \tag{3.4}$$

$$-2 \sum_{w,z|w^0+1/2\le p\le z^0-1/2} \Gamma(u,w) \sum_{r|r^0+1/2=p} G(w,r)G(r+e^0,z)\,\Gamma(z,v)\Big|_{\kappa_p=0} .$$

Using Lemma 3.1 (first item) for $r=0$ the w and z sums can be taken over all values to give the result. The third statement follows from a similar argument. The fourth one follows from the first three and using Eq. (3.3) again. □

We are now ready to prove Theorem 2.1 .

Proof of Theorem 2.1 (first item). Consider the expansion of the denominator of $G_{\alpha\beta}$ in powers of κ. For a point where the bonds arrive and leave in opposite directions the Fermi integration gives products of Γ's, in which case the above property is used to give zero. The first non-vanishing contribution occurs at κ^8, corresponding to two sets of four bonds going around an elementary square in opposite directions. For the numerator we explicitly carry out two typical calculations: one for the κ^2 contribution to $G_{11}(x)$, associated with $x = \epsilon e^\rho$ ($|x| = 1$), and another for the case of an angle contribution to $G_{11}(x)$, when $x = \epsilon e^\rho + \epsilon' e^\sigma$. The case $\epsilon = \epsilon'$ and $\rho = \sigma$ is simpler as the property $\Gamma^{\epsilon e^\rho}\Gamma^{\epsilon e^\rho} = -2\Gamma^{\epsilon e^\rho}$ can be used.

The κ^2 contribution to the case $|x = u - v| = 1$, involves two bonds, with opposite orientation connecting u and v. After performing the gauge group integral, we obtain for $G_{11}(u,v)$ with $\langle\ \rangle_0$ denoting the expectation with the hopping parameter set equal to zero in the action of (2.2)

$$\frac{\kappa^2}{12}\langle\mu_1(u)\bar\psi_{a_1\alpha_1}(u)\psi_{a_1\beta_2}(u)\rangle_0\Gamma^{\epsilon e^\rho}_{\alpha_1\beta_1}\Gamma^{-\epsilon e^\rho}_{\alpha_2\beta_2}\langle\psi_{a_2\beta_1}(v)\bar\psi_{a_2\alpha_2}(v)\pi_1(v)\rangle_0.$$

Using the definitions (2.3), (2.4) and applying again Wick's theorem for the factors $\langle\ \rangle_0$ we get

$$\frac{\kappa^2}{8}\left(\Gamma^{-\epsilon e^\rho}_{11}\Gamma^{\epsilon e^\rho}_{33} + \Gamma^{-\epsilon e^\rho}_{12}\Gamma^{\epsilon e^\rho}_{43} + \Gamma^{-\epsilon e^\rho}_{21}\Gamma^{\epsilon e^\rho}_{34} + \Gamma^{-\epsilon e^\rho}_{22}\Gamma^{\epsilon e^\rho}_{44}\right).$$

By the explicit structure of the matrices Γ the second and third terms above are zero (for all ρ) and the sum of the first and fourth is equal to $8\delta_{\rho 0}+2\delta_{\rho j}$ (remember that $j=1,2,3$) and the result follows.

The κ^4 contribution to $G_{11}(u,v) = G_{11}(x=u-v)$ of the angle $0\to\epsilon e^\rho \to \epsilon e^\rho+\epsilon' e^\sigma \equiv x$ is denoted by $A_{11}^{\epsilon e^\rho,x}\kappa^4$ and has $g_{0,\epsilon e^\rho}$ $(g^{-1}_{0,\epsilon e^\rho})$ emanating from 0 (ϵe^ρ) in the e^ρ direction and $g_{\epsilon e^\rho,x}$ $(g^{-1}_{\epsilon e^\rho,x})$ arriving at x (ϵe^ρ) in the e^σ direction. After gauge integration of $g_{0,\epsilon e^\rho}g_{\epsilon e^\rho,0}$ and $g_{\epsilon e^\rho,x}g_{x,\epsilon e^\rho}$, and Fermi integration of the fields at ϵe^ρ using Wick's theorem, we have

$$\begin{aligned} A_{11}^{\epsilon e^\rho,x} \;=\;& \frac{1}{48}\langle\mu_1(u)\bar\psi_{a_1\alpha_1}(u)\psi_{a_1\beta_2}(u)\rangle_0\,\Big(\Gamma^{\epsilon e^\rho}_{\alpha_1\beta_1}\Gamma^{-\epsilon e^\rho}_{\alpha_2\beta_2}\Gamma^{\epsilon' e^\sigma}_{\beta_1\beta_3}\Gamma^{-\epsilon' e^\sigma}_{\alpha_4\alpha_2} \\ & -3\Gamma^{\epsilon e^\rho}_{\alpha_1\beta_1}\Gamma^{-\epsilon e^\rho}_{\beta_1\beta_2}\Gamma^{\epsilon' e^\sigma}_{\alpha_3\beta_3}\Gamma^{-\epsilon' e^\sigma}_{\alpha_4\alpha_3}\Big)\,\langle\psi_{a_4\beta_3}(v)\bar\psi_{a_4\alpha_4}(v)\pi_1(v)\rangle_0\,. \end{aligned}$$

The second product of Γ's above is zero by $\Gamma^{\epsilon e^\rho}\Gamma^{-\epsilon e^\rho}=0$. Using the definitions (2.3), (2.4) and applying again Wick's theorem for the factors $\langle\;\rangle_0$ we get

$$\begin{aligned} A_{11}^{\epsilon e^\rho,x} \;=\;& \frac{1}{32}\Big[\Lambda_{11}^{-\epsilon' e^\sigma,-\epsilon e^\rho}\Lambda_{33}^{\epsilon e^\rho,\epsilon' e^\sigma}+\Lambda_{12}^{-\epsilon' e^\sigma,-\epsilon e^\rho}\Lambda_{43}^{\epsilon e^\rho,\epsilon' e^\sigma} \\ & +\Lambda_{21}^{-\epsilon' e^\sigma,-\epsilon e^\rho}\Lambda_{34}^{\epsilon e^\rho,\epsilon' e^\sigma}+\Lambda_{22}^{-\epsilon' e^\sigma,-\epsilon e^\rho}\Lambda_{44}^{\epsilon e^\rho,\epsilon' e^\sigma}\Big] \end{aligned}$$

where $\Lambda_{uu'}^{-\epsilon' e^\sigma,-\epsilon e^\rho}\equiv\Gamma^{-\epsilon' e^\sigma}_{u\alpha}\Gamma^{-\epsilon e^\rho}_{\alpha u'}$ and $\Lambda_{\ell\ell'}^{\epsilon e^\rho,\epsilon' e^\sigma}\equiv\Gamma^{\epsilon e^\rho}_{\ell\beta}\Gamma^{\epsilon' e^\sigma}_{\beta\ell'}$. Similarly, we can calculate the contribution of $A_{11}^{\epsilon' e^\sigma,x}$. The expression for $c_{11}(x)$ is

$$c_{11}(x) \;=\; A_{11}^{\epsilon e^\rho,x}+A_{11}^{\epsilon' e^\sigma,x},$$

with $x=\epsilon e^\rho+\epsilon' e^\sigma$, and similarly for the other $c_{\alpha\beta}(x)'s$. When $\rho=i$ and $\sigma=j$ $(i,j=1,2,3)$ by the symmetry properties of Theorem 4 in [17] (namely, rotations by π about e^3 and parity), we can show that $c_{\alpha\beta}(x=\epsilon e^i+\epsilon' e^j)$ is independent of ϵ, ϵ' so, we can write $c_{\alpha\beta}(x)=c_{\alpha\beta}(e^i+e^j)\equiv c^{ij}_{\alpha\beta}$ and we get Eq. (2.10). □

Proof of Theorem 2.1 (second item). Using a Cauchy integral representation for each κ_p and for analogous spatial complex hyperplane decoupling parameters, taking into account the number of vanishing derivatives as given in Lemma 3.1 , and using Cauchy estimates on the multiple integral gives the result. An argument based on the maximum modulus theorem could also be used (see [9] for details). □

Let us now turn to the proof of Theorem 2.2 . The corrections to the asymptotic mass value of $-2\ln\kappa$ which we need for the determination of mass splitting require precise values of $\Gamma_{\alpha\beta}(x)$, for small $|x|$. The results go beyond those obtained by the hyperplane method, and rely on explicit cancellations in the Neumann series for Γ. The results below are obtained expanding in powers of κ and controlling the remainders using the analyticity of G and Γ, and the decay of G as given by Theorem 2.1 .

Proof of Theorem 2.2 (first item). $\Gamma_{\alpha\beta}$ is obtained from the Neumann series and Theorem 2.1 (first item). We show how the cancellations occur, which improves the hyperplane method bound. We explicitly consider the case $x=\epsilon e^0+\epsilon' e^j$; the other cases where there are one, two or three times units are treated similarly. Recall that [see Eq. (2.8)] $G_{d,\alpha\beta}(u,v)=G_{\alpha\alpha}(u,u)\delta_{\alpha\beta}\delta(u,v)$. From Theorem 2.1 , we obtain $G_{\alpha\alpha}(0)=1+\mathcal{O}(\kappa^8)$. Using $\Gamma=\sum_{i=0}^{\infty}(-1)^i[G_d^{-1}G_n]^iG_d^{-1}$, for $x=u-v\neq 0$, we write

$$\begin{aligned}\Gamma_{\alpha\beta}(u,v) &= -G^{-1}_{\alpha\alpha}(0)G_{n,\alpha\beta}(u,v)G^{-1}_{\beta\beta}(0) + \sum_w G^{-1}_{\alpha\alpha}(0)G_{n,\alpha\gamma}(u,w)\\ &\quad \times G^{-1}_{\gamma\gamma}(0)G_{n,\gamma\beta}(w,v)G^{-1}_{\beta\beta}(0) + \mathcal{O}(G_n^3). \qquad (3.5)\end{aligned}$$

For $\alpha=\beta$, there are two κ^4 angle contributions to $G_{n,\alpha\beta}(u,v)$ in the first term of Eq. (3.5), and these are cancelled by the product of κ^2 contributions for $w-v=\epsilon e^0$ and $w-v=\epsilon e^j$ in the second term of Eq. (3.5). □

Proof of Theorem 2.2 (2nd item). Similar to Theorem 2.1 (2nd item). □

4 Spectral Results

We now prove Theorem 2.3 . To determine the meson masses and dispersion curves, we find the p^0 imaginary axis solutions of $\det\tilde{\Gamma}(p^0,\vec{p})=0$. For the mass spectrum, we find, by the use of symmetries, that $\tilde{\Gamma}_{\alpha\beta}(p^0,\vec{p}=\vec{0})$ is diagonal. Furthermore, we show that $m_c+2\ln\kappa$ is real analytic in κ. For $\vec{p}\neq\vec{0}$, as we have *not* found symmetries which simplify the matrix structure, we determine the dispersion curves $w(\vec{p})$, where $\det\tilde{\Gamma}(p^0=iw(\vec{p}),\vec{p})=0$, by an application of Rouché's theorem.

We state some symmetry properties in the next lemma.

Lemma 4.1 *The following symmetry properties are satisfied by the matrices G and Γ.*

1. $G_{\alpha\beta}(x)=[G_{\beta\alpha}(x)]^*$ *and* $\Gamma_{\alpha\beta}(x)=[\Gamma_{\beta\alpha}(x)]^*$;
2. *For* $\chi\in\mathbb{R}$, *let* $p^0=i\chi$. *We have* $\tilde{G}_{\alpha\beta}(i\chi,\vec{p})=[\tilde{G}_{\beta\alpha}(i\chi,\vec{p})]^*$ *and* $\tilde{\Gamma}_{\alpha\beta}(i\chi,\vec{p})=[\tilde{\Gamma}_{\beta\alpha}(i\chi,\vec{p})]^*$;
3. *At* $\vec{p}=\vec{0}$, $\tilde{G}_{\alpha\beta}(p^0,\vec{p}=\vec{0})=\mathrm{diag}[a+c,b,b,a-c]$, *with* $a,b,c\in\mathbb{C}$.

Proof . Here we use several symmetries given in Theorem 4 in [17]. By the spectral representation of Proposition 2.1 , for $x^0\neq 0$, item one follows. For $x^0=0$, the result follows from time reversal and parity. Thus, item one holds for all x. The proof of item two follows from parity invariance $G_{\alpha\beta}(x^0,\vec{x})=G_{\alpha\beta}(x^0,-\vec{x})$. To prove item three, we use reflection symmetry in the coordinate x^1. Here $\vec{x}=(x^1,\ldots,x^d)\mapsto\vec{x}\,'=(-x^1,\ldots,x^d)$, and $\tilde{\psi}_{a1}(x)\mapsto-\tilde{\psi}_{a2}(x')$, $\tilde{\psi}_{a2}(x)\mapsto-\tilde{\psi}_{a1}(x')$, $\tilde{\psi}_{a3}(x)\mapsto\tilde{\psi}_{a4}(x')$, $\tilde{\psi}_{a4}(x)\mapsto\tilde{\psi}_{a3}(x')$. Also rotations of

$\pi/2$ and π about e^3 will be used. Considering $\pi/2$ and $d=3$ we have, $x=(x^0,x^1,x^2,x^3)\mapsto x'=(x^0,-x^2,x^1,x^3)$ $[x=(x^0,x^1,x^2)\mapsto x'=(x^0,-x^2,x^1)$, for $d=2]$ and $\tilde\psi_{a1}(x)\mapsto\tilde\psi_{a1}(x')$, $\psi_{a2}(x)\mapsto -i\psi_{a2}(x')$, $\tilde\psi_{a3}(x)\mapsto\tilde\psi_{a3}(x')$, $\psi_{a4}(x)\mapsto -i\psi_{a4}(x')$ and $\bar\psi_{a2}(x)\mapsto i\bar\psi_{a2}(x')$, $\bar\psi_{a4}(x)\mapsto i\bar\psi_{a4}(x')$. For π and $d=3$ we have, $x=(x^0,x^1,x^2,x^3)\mapsto x'=(x^0,-x^1,-x^2,x^3)$ $[x=(x^0,x^1,x^2)\mapsto x'=(x^0,-x^1,-x^2)$, for $d=2]$ and $\tilde\psi_{a1}(x)\mapsto\tilde\psi_{a1}(x')$, $\tilde\psi_{a2}(x)\mapsto-\tilde\psi_{a2}(x')$, $\tilde\psi_{a3}(x)\mapsto\tilde\psi_{a3}(x')$, $\tilde\psi_{a4}(x)\mapsto-\tilde\psi_{a4}(x')$.

These transformations, taken together with the symmetry results given in Theorem 4 in [17], are used to obtain the matrix structure stated in the third item, for $d=2,3$. □

After using Theorem 2.2 and taking the Fourier transform, we have

$$\begin{aligned}\tilde\Gamma_{\alpha\beta}(p^0,\vec 0) \;=\;& \left[1-2dc_2\kappa^2+2(1-dc_2-12c_2^2+10dc_2^2)\kappa^4\right]\delta_{\alpha\beta}-4(c^{12}_{\alpha\beta}+\\ & c^{13}_{\alpha\beta}+c^{23}_{\alpha\beta})\kappa^4+\mathcal{O}(\kappa^6)-\left[\delta_{\alpha\beta}\kappa^2+\mathcal{O}(\kappa^8)\right](e^{ip^0}+e^{-ip^0})+\ldots,\end{aligned} \tag{4.1}$$

We now introduce an auxiliary function $H_\alpha(w,\kappa)$, jointly analytic in w and κ, for small $|\kappa|$ and $|w|$, such that $H_\alpha(w=1-\kappa^2e^{-ip^0},\kappa)=\tilde\Gamma_{\alpha\alpha}(p^0,\vec p=\vec 0)$. $H_\alpha(w,\kappa)$ is defined by

$$\begin{aligned}H_\alpha(w,\kappa) \;=\;& w-2dc_2\kappa^2+2(1-dc_2-12c_2^2+10dc_2^2)\kappa^4-4(c^{12}_{\alpha\alpha}+c^{13;23}_{\alpha\alpha})\kappa^4\\ & -\frac{\kappa^4}{1-w}+\sum_{\vec x}\Gamma'_{\alpha\alpha}(0,\vec x)-\sum_{\vec x}\Gamma'_{\alpha\alpha}(1,\vec x)\left[\frac{\kappa^2}{1-w}+\frac{1-w}{\kappa^2}\right]+\\ & \sum_{n\ge1,\vec x|(n,\vec x)\ne(1,\vec 0)}\Gamma_{\alpha\alpha}(n,\vec x)\left[\left(\frac{\kappa^2}{1-w}\right)^n+\left(\frac{1-w}{\kappa^2}\right)^n\right],\end{aligned} \tag{4.2}$$

where $\Gamma'_{\alpha\alpha}(0,\vec x)$ (respectively, $\Gamma'_{\alpha\alpha}(1,\vec x)$) contains the contributions of $\mathcal{O}(\kappa^6)$ (respectively, $\mathcal{O}(\kappa^8)$) or higher and $c^{13;23}_{\alpha\alpha}$ is the sum of angle contributions $0\to e^i\to e^{13}$ $(i=1,3)$ and $0\to e^j\to e^{23}$ $(j=2,3)$ which is diagonal. $H_\alpha(0,0)=0$ and $\frac{\partial H_\alpha}{\partial w}(0,0)=1$, and hence the analytic implicit function theorem applies and gives us an analytic function $w(\kappa)\equiv w_\alpha(\kappa)$ such that $H_\alpha(w(\kappa),\kappa)=0$. Thus, for κ real positive, the mass is given by

$$m_\alpha=-\ln\kappa^2+\ln(1-w)\;.$$

By an analysis of the formulas for the implicit function derivatives, with $dw=-(\partial_wH)^{-1}\partial_kH\equiv-(H_w)^{-1}H_k$, we find $d_0^rw=0$, $r=0,1,3,5$ and

$$\begin{aligned}d_0^2w \;&=\; -\partial_\kappa^2H(0,0)\;=\;4c_2d\\ d_0^4w \;&=\; -\partial_\kappa^4H(0,0)\;=\;-4!\left[1-2dc_2-24c_2^2+20dc_2^2-4(c^{12}_{\alpha\alpha}+c^{13;23}_{\alpha\alpha})\right].\end{aligned}$$

Hence, for $d=3$,

$$\begin{aligned}m_\alpha \;=\;& -2\ln\kappa-2dc_2\kappa^2+\\ & \left[-4(c^{12}_{\alpha\alpha}+c^{13;23}_{\alpha\alpha})+(1-2dc_2-24c_2^2+20dc_2^2)-2d^2c_2^2\right]\kappa^4+\mathcal{O}(\kappa^6)\,,\end{aligned}$$

and similarly, for $d=2$, with $c^{13;23}_{\alpha\alpha}=0$.

Thus, up to and including $\mathcal{O}(\kappa^4)$, we have, for $d=3$, $m_a\equiv m_\alpha$ and $m_b\equiv m_1$, $m_a-m_b=2\kappa^4$ and $\alpha=2,3,4$; for $d=2$, $m_a\equiv m_\alpha$ and $m_b\equiv m_\beta$, $m_a-m_b=\kappa^4$, $\alpha=2,3$ and $\beta=1,4$.

Let us now turn to the dispersion curves. They satisfy $\det\tilde\Gamma(ip^0=iw(\vec p),\vec p)=0$. To determine them, with $c_2(\vec p)\equiv c_2\sum_{j=1}^d 2\cos p^j$, we write the Fourier transform of $\Gamma_{\alpha\beta}(x)$ as

$$\begin{aligned}\tilde\Gamma_{\alpha\beta}(p^0,\vec p) &= \left[1-c_2(\vec p)\kappa^2-\kappa^2(e^{-ip^0}+e^{ip^0})\right]\delta_{\alpha\beta}\\ &\quad+\sum_{n,\vec x}{}'\;\Gamma_{\alpha\beta}(n,\vec x)\,e^{-i\vec p.\vec x}(e^{-ip^0n}+e^{ip^0n})\,, \end{aligned}\tag{4.3}$$

where $\sum_{n,\vec x}{}'$ means that all terms of order κ^4 or higher in $\Gamma_{\alpha\beta}(x)$ are included. Introduce the auxiliary matrix function $H_{\alpha\beta}(w,\kappa)\equiv H_{\alpha\beta}(w,\kappa,\vec p)$ such that $H_{\alpha\beta}(w=1-c_2(\vec p)\kappa^2-\kappa^2e^{-ip^0},\kappa)=\tilde\Gamma_{\alpha\beta}(p^0,\vec p)$. $H_{\alpha\beta}(w,\kappa)$ is defined by

$$\begin{aligned}H_{\alpha\beta}(w,\kappa) &= w\delta_{\alpha\beta}+\sum_{n,\vec x}{}''\;\Gamma_{\alpha\beta}(n,\vec x)\,e^{-i\vec p.\vec x}\;\times\\ &\quad\left[\left(\frac{1-w-c_2(\vec p)\kappa^2}{\kappa^2}\right)^n+\left(\frac{\kappa^2}{1-w-c_2(\vec p)\kappa^2}\right)^n\right],\end{aligned}$$

where $\sum_{n,\vec x}{}''$ means that only $\mathcal{O}(\kappa^4)$ terms or higher order terms are to be included. $H(w,\kappa)$ is jointly analytic in κ and w at $(w,\kappa)=(0,0)$.

Letting

$$\begin{aligned}f(w)\;\equiv\;\det H(w,\kappa) &= \det wI_4\;+\;[\det H(w,\kappa)-\det wI_4]\\ &\equiv g(w)\;+\;h(w)\,,\end{aligned}$$

we can apply Rouché's theorem to $f(w)$ on the circle $|w|=c|\kappa|^4$, $c\ggg 1$, $|g(w)|=c^4|\kappa|^{16}$ and $|h(w)|\le c'|\kappa|^{16}<c^4|\kappa|^{16}=|g(w)|$, so that $f(w)$ has four zeroes inside $|w|=c|\kappa|^4$, as $g(w)=w^4$ has a fourth order zero. Notice that the upper bound for $|h(w)|$ comes from an upper bound on the remaining twenty three terms in the difference between the two determinants. Now, for $p^0=iw(\vec p)$, and κ real positive, each of the four zeroes satisfying $\det\tilde\Gamma(p^0=iw(\vec p),\vec p)=0$ has the form

$$w(\vec p)\quad=\quad-2\ln\kappa-2dc_2\kappa^2+2c_2\kappa^2\textstyle\sum_{j=1}^d(1-\cos p^j)+\mathcal{O}(\kappa^4)\,.$$

We now extend the spectral results from $\mathcal{H}_m$ to the space $\mathcal{H}_e$, using the Euclidean subtraction method established in [8]. We consider the generalized subtracted two-point cf

$$\mathcal{F}(u,v)\quad=\quad\mathcal{G}_{LL}(u,v)-\sum_{w,w'\in\mathbb{Z}_0^{d+1}}\mathcal{G}_{L,\Phi}(u,w)\Gamma(w,w')\mathcal{G}_{\Phi,L}(w',v)\,,$$

where $\Phi(u) = (\pi_1(u), ..., \pi_4(u))$ has four components; $\Gamma(w, w')$ is given by the convolution inverse of the two-point function G. Finally,

$$\mathcal{G}_{FH}(u,v) = \begin{cases} S_{\Theta T_0^{-1} F, H}(u,v)\,,\; u^0 \leq v^0 \\ S^*_{F, \Theta T_0^{-1} H}(u,v)\,,\; u^0 > v^0 \end{cases}$$

with $S_{F,H}(u,v) = \langle [T_0^{u^0-1/2} \vec{T}^{\vec{u}} F(1/2,\vec{0})][T_0^{v^0-1/2} \vec{T}^{\vec{v}} H(1/2,\vec{0})]\rangle_T$.

The lemma below guarantees that our results hold in the full space $\mathcal{H}_e$.

Lemma 4.2 *For $u^0 < v^0$, $p \in \mathbb{Z}$, $u^0 + 1/2 \leq p \leq v^0 - 1/2$ (or, if $u^0 > v^0$, $v^0 + 1/2 \leq p \leq u^0 - 1/2$), and again denoting by ∂_0 the κ_p derivative at $\kappa_p = 0$, we have $\partial_0^r \mathcal{F}(u,v) = 0$, for $r = 0, 1, 2, 3$.*

Proof . The proof follows closely [8]. For $u^0 < v^0$ (the case $u^0 > v^0$ is similar) and from Lemmas 3.1 and 3.2 the power series expansions in κ_p of the functions appearing in the definition of $\mathcal{F}(u,v)$ are of the form

$$\begin{aligned}
\mathcal{G}_{LL}(u,v) &= a_2(u,v)\kappa_p^2 + \mathcal{O}(\kappa_p^4)\,, \\
\mathcal{G}_{L,\Phi}(u,w) &= b_0(u,w)\chi_{w^0 \leq -1/2+p} + b_2(u,w)\kappa_p^2 + \mathcal{O}(\kappa_p^4)\,, \\
\Gamma(w,w') &= c_0(w,w')(\chi_{w^0 \leq -1/2+p}\,\chi_{w'^0 \leq -1/2+p} + \chi_{w^0 > -1/2+p} \\
&\quad \times \chi_{w'^0 > -1/2+p}) + c_2(w,w')\kappa_p^2 + \mathcal{O}(\kappa_p^4)\,, \\
\mathcal{G}_{\Phi,L}(w',v) &= d_0(w',v)\chi_{w'^0 > -1/2+p} + d_2(w',v)\kappa_p^2 + \mathcal{O}(\kappa_p^4)\,.
\end{aligned}$$

Substituting these expressions, it is easy to show that $\partial_0^r \mathcal{F}(u,v) = 0$ ($r = 0, 1, 3$). For $r = 2$, proceeding exactly as in Theorem 3.7 of [8], we have four terms which sum up to zero and the result follows. □

This ends the proof of Theorem 2.3 . □

5 Acknowledgements

We warmly thank Ricardo Schor for useful discussions and the anonymous referee for suggestions that improved the paper. This work was supported by CNPq and Pronex. A.F. Neto acknowledges the CNPq for a doctoral scholarship.

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